1. Introduction

Here is presented calculation methods for single-pylon suspension structures, stiffened by girders (Figure 1).

The common assumptions about the linear elastic strain-stress dependence of materials and absence of elongations of hangers are taken into account. The cables are regarded as geometrically nonlinear rods without bending rigidity, and stiffening girder – as bended linear bar. The mutual action between the carrying cables and stiffening members are regarded as a nodal contact load in supporting nodes of hangers.

Discrete analysis is based on condition of equilibriums, made for the nodal points of the cable. Under uniformly distributed load the cable will take parabolic form. In reality, the cable is loaded by concentrated forces and it takes the form of a string polygon. The conditions of equilibrium are written for every node of polygon and elongation of the cable was determined using equation of deformations compatibility for every section of the cable. These conditions form nonlinear

Fig. 1. The schemes of the single-pylon suspension bridges stiffened by a girder

**DISCRETE ANALYSIS FOR SINGLE-PYLON SUSPENSION BRIDGES**

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**Abstract.** This paper presents calculation methods, developed at Tallinn University of Technology (Idnurm 2004; Kulbach 2007), for the single-pylon suspension bridges stiffened by a girder. Classical suspension bridge consists of a geometrically non-linear cable, connected by hangers with an elastic linear stiffening girder, pylons in both ends of the bridge and anchor cables. Alternate form of a suspension bridge is a bridge, with only one pylon on the middle of the span and suspension cable is connected to the abutments or the ends of the stiffening girder. In the calculation of suspension bridges, the geometrically non-linear behaviour of the parabolic cable is the main problem. The linear methods of analysis suit only for small spans. A geometrically non-linear continual model is especially useful for classical loading cases – a uniformly distributed load on the whole or half span. But the modern traffic models consist of concentrated and uniformly distributed loads. The discrete model of a suspension bridge allows us to apply all kinds of loads, such as distributed or concentrated ones. The assumptions of the discrete method described here are: linear elastic strain-stress dependence of the material and absence of horizontal displacements of hangers. Hanger elongation is taken into account.

**Keywords:** cable structure, girder-stiffened structure, suspension bridge, long-span structures, discrete analysis, geometric non-linearity.
equation system, which give after solving all nodes displacements and internal forces for cables and for stiffening girder.

Here is used the scheme described as follows:
1. The initial balance of the cable (before the following loading) is described.
2. The final balance of the cable is described (the non-connected pieces of the stiffening girder are hanging on the cable and the cable is deformed).
3. The final balance of the whole structure is described (the stiffening girder is consistent and in initial solution takes approximately 10% of all load, 90% of load is imposed to the cable).

2. Basic equations

2.1. Discrete model for elastic cable

The initial state of equilibrium of the cable loaded by a concentrated load is shown in Figure 2.

From the equilibrium considerations of forces we may write for every node (Kulbach, Ogier 1986)

\[
F_{0,i} + H_0 \frac{z_{0,i+1} - z_{0,i}}{a_{0,i}} + H_0 \frac{z_{0,i-1} - z_{0,i}}{a_{0,i-1}} = 0, \tag{1}
\]

where \(F_{0,i}\) – initial external force; \(H_0\) – initial force horizontal component of the cable (cable force); \(z_{0,i-1}, z_{0,i}, z_{0,i+1}\) – initial vertical coordinates of the cable nodes; \(a_{0,i-1}, a_{0,i}\) – horizontal distance between hangers.

For a cable which has supporting nodes on different level we may calculate \(H_0\) as (Kulbach, Ogier 1986)

\[
H_0 = \frac{a_{0,0} \sum_{i=1}^{n} F_{0,i} (L_0 - x_{0,i})}{L_0 (z_{0,1} - z_{0,0}) + a_{0,0} (z_{0,0} - z_{0,n+1})}, \tag{2}
\]

where \(L_0\) – span of the cable.

By the action of the temporary loads \(\Delta F_i\) (Figure 3), the equilibrium equation for the node \(i\) is expressed as (Kulbach, Ogier 1986)

\[
F_i + H \frac{z_{0,i+1} - z_{0,i}}{a_{0,i}} + H \frac{z_{0,i-1} - z_{0,i}}{a_{0,i-1}} = \frac{w_i}{a_{0,i}}, \tag{3}
\]

where \(F_i\) – final external nodal forces \((F_i = F_{0,i} + \Delta F_i)\); \(H\) – thrust from temporary and initial load; \(w_{i-1}, w_i, w_{i+1}\) – vertical displacements of the nodes.

There are two unknown parameters in Eq. (3): \(w_i\) and \(H\). Thus we need another equation for calculating them. We can use the compatibility condition of the relative elongation of the cable. The relative elongation of the cable is expressed as (Kulbach, Ogier 1986)

\[
\varepsilon_i = \frac{1}{1 + \left( \frac{z_{0,i+1} - z_{0,i}}{a_{0,i}} \right)^2} \left[ \frac{u_{i+1} - u_i}{a_{0,i}} + \frac{w_{i+1} - w_i}{a_{0,i}} \left( \frac{z_{0,i+1} - z_{0,i}}{a_{0,i}} + \frac{w_{i+1} - w_i}{2a_{0,i}} \right) \right]. \tag{4}
\]
and from condition of linear deformation
\[ e_i = \frac{H - H_0}{EA} \sqrt{1 + \left( \frac{z_{0,i+1} - z_{0,i}}{a_{0,i}} \right)^2}, \quad (5) \]
where \( EA \) – stiffness of the cable in tension; \( u_{i-1}, u_i, u_{i+1} \) – horizontal displacements of the nodes.

Taking into account (4) and (5), this compatibility condition may be presented as (Kulbach, Öiger 1986)
\[ \sum_{i=0}^{n} (u_{i+1} - u_i) = u_{n+1} - u_0. \quad (7) \]
where \( u_{n+1}, u_0 \) – horizontal displacements of the support nodes of the cables.

We may write the Eq. (6) in the form (Kulbach, Öiger 1986)
\[ \sum_{i=0}^{n} \left( \frac{z_{0,i+1} - z_{0,i}}{a_{0,i}} \right)^2 = \frac{H - H_0}{EA} \left( \frac{u_{n+1} - u_0}{H - H_0} \right). \quad (8) \]

Horizontal displacements of the internal nodes may be eliminated by means of summation of the equations of deformation compatibility (6) and after replacing (Kulbach, Öiger 1986)
\[ \sum_{i=0}^{n} \left( \frac{z_{0,i+1} - z_{0,i}}{a_{0,i}} \right)^2 = \frac{H - H_0}{EA} \left( \frac{u_{n+1} - u_0}{H - H_0} \right). \]
\[ \sum_{i=1}^{n} \left( \frac{w_{i+1} - w_i}{a_{0,i}} \right)^2 = \frac{H - H_0}{EA} \left( \frac{u_{n+1} - u_0}{H - H_0} \right). \]

In case of single-pylon suspension bridge, we have two spans, which are carried by the main cable. Because we have two spans, we must calculate two different girder-stiffened cable systems, which are connected to the central pylon. Then, for the first span, we have horizontal displacements for cable ends: \( u_{0,1} = 0; u_{n+1,1} \) is unknown. For the second span, \( u_{0,2} = u_{n+1,1}; u_{n+1,0} = 0. \)

For initial form of the bridge, when a bridge is symmetrical, we can obtain \( H_I = H_{II} \). In loaded state, the central pylon has horizontal deformation and if the bottom support of pylon is fixed, then the final cable forces \( H_I \neq H_{II} \); if bottom support of the pylon has released, then \( H_I = H_{II} \) (Figure 4). The relations between \( H \) and \( H_{II} \) can be described, when bending rigidity of pylons is constant, as (Idnurm, J., Kiisa, Idnurm, S. 2009)

\[ \frac{H_{II}^3 h_p^3}{6} - \frac{H_I^3 h_p^3}{6} - E_p I_p u_p = 0. \quad (9) \]
where \( h_p \) – height of the central pylon; \( E_p I_p \) – pylon’s bending rigidity; \( u_p \) – horizontal displacement in top of the pylon.

![Fig. 4. The calculation scheme of the cable internal forces depending on the characterization of the pylon](image)

Single pylon suspension bridge has two unknown cable forces \( H_I, H_{II} \) and one unknown horizontal displacement \( u_p \). For this kind of bridge, we can calculate all displacements of internal nodes of both cable and contact forces in hangers, if \( H_I, H_{II} \) and \( u_p \) is determined. The best way is to take initial displacement \( u_p = 0 \) and solve this system is to calculate both spans separately. After solving the systems, we can calculate new value of \( u_p \) using condition (9), and repeat this process until change of pylon top displacement \( \Delta u_p = 0 \). If bottom support of the pylon has released, then we can’t calculate \( u_p \) directly, but then we can search the value of \( u_p \), to find condition \( H_I = H_{II} \).

2.2. Analysis of the stiffening girder

The different schemes of a girder are presented in Figures 5–7. Let us consider only the stiffening girder. The equation that describes the deflection of the girder can be written as follows (Jürgenson 1985):
\[ w(x) = w_0 + x - \sum_{j=1}^{m} \frac{M_j}{2 E_b I_b} (x - x_{M,j})^2 \]
\[ + \sum_{k=1}^{n} \frac{x - x_{P,k}}{6 E_b I_b} + \sum_{l=1}^{d} \frac{x - x_{P,l}}{24 E_b I_b} \]
\[ + \sum_{j=1}^{4} \frac{x - x_{P,j}}{24 E_b I_b} \]
where $w_0$ – initial deflection of rotation at the first point of the girder; $\varphi_0$ – angle of rotation at the first point of the girder; $M_j$ – concentrated moments; $P_k$ – concentrated forces (internal forces in the hangers, vertical support reactions and external loads); $p_l$ – uniformly distributed loads; $E_b I_b$ – rigidity of the stiffening girder in bending; $\mathcal{H}(x)$ – the Heaviside’s function.

Equation (10) can be used for calculating deflection from the sum of applied external concentrated moments, concentrated forces and from the uniformly distributed loads. The constants $w_0$ and $\varphi_0$ are all different on schemes described in Figures 5–7. They can be calculated using Equation (10) in points, where displacements are known (on supports).

2.3. Discrete model for the girder stiffened cable

There are different solutions described to calculate the stiffening girder and the cable as an integrated construction (Idnurm 2004). The new solution plan is defined as follows:

1. The cable and the stiffening girder can be calculated separately. The final solution is found when vertical displacements of the cable and stiffening girder are the same in points which are connected with the same hanger.
2. The first step is to calculate the cable initial forces $H_0$ and the shape of the cable $z_i$ using only self-loads (the non-connected blocks of the stiffening girder are hanging on the cable).
3. 10% of the successive load is imposed to the girder and vertical displacements $w_i$ of the points where hangers are connected will be found (the contact forces of the hangers are not taken into account).
4. The cable initial forces $H_I$ and $H_{II}$ can be found in loaded condition.
5. The inside forces of the hangers can be calculated using the equilibrium considerations of forces of cable nodes.
6. Vertical displacements $w_i$ of the points where hangers are connected will be found again using all loads placed on the girder and also the contact forces of the hangers.
7. The final step is to compare displacements found in subsection 6 with displacements found in subsection 3 using the next formula: 
   \[ \sum_{i=1}^{n} (w_{i,new} - w_{i,old})^2 \]. If this sum total is near the 0, then calculated inside forces and displacements are final and solution is finished. If the sum total is not 0, the calculation process must be repeated from the subsection 3 using loads which are corrected by comparison with mentioned 10%.

3. Numerical results

Comparison of results for classical suspension bridge from different calculation methods are presented in
(Idnurm 2004; Kulbach, Idnurm, J., Idnurm, S. 2002; Kulbach 1995, 1998, 1999, 2007). There the results from analytical (continual) method (Aare, Kulbach 1984) and discrete method give similar results, which are more exact than results, got from linear finite element method (because of finite element method defines the cable as many single segments which are connected).

For the numerical results we used single-pylon suspension bridge with spans 90 m. Height of the pylon is taken 20 m.

In the calculation, three different calculation methods were used: discrete methodology, linear finite element method (FEM), non-linear FEM. For FEM is used commercial software Staad/Pro v8i. The loads and basis of modelling using FEM are the same as for the analytical (continual) method and discrete method.

The calculations were carried out with the combinations of the cross-sections of two different cables, with cross-section areas 31800 mm$^2$ and 45800 mm$^2$. The modulus of elasticity for the cable was taken 170 GPa, for the hangers and girder the modulus of elasticity was 210 GPa. The moment of inertia of the girder was taken $3.38 \times 10^{10}$ mm$^4$.

The prestressing load, which consist self-weight of the girder and deck structure is $p_0 = 30$ kN/m (at that moment there are no deflections of the girder and the cable because the cable is prestressed), additional self-weight load is $p_1 = 20$ kN/m, and traffic load $p_t = 30$ kN/m.

The geometry and loads for bridge is presented in Figure 8. Total load $p$ for this calculation is taken between $p_{20} = 20$ kN/m to $p_{50} = 50$ kN/m. Then, maximum displacement from traffic load $p_t = 30$ kN/m can be calculated $w_t = w$ from $p_{50} - w$ from $p_{20}$.

Table 1. Maximum deflections of the girder, m

<table>
<thead>
<tr>
<th>$p$, kN/m</th>
<th>$A = 31800$ mm$^2$</th>
<th>$A = 45800$ mm$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Discrete</td>
<td>Linear</td>
</tr>
<tr>
<td>20</td>
<td>-0.193</td>
<td>-0.215</td>
</tr>
<tr>
<td>30</td>
<td>-0.238</td>
<td>-0.323</td>
</tr>
<tr>
<td>40</td>
<td>-0.370</td>
<td>-0.430</td>
</tr>
<tr>
<td>50</td>
<td>-0.454</td>
<td>-0.538</td>
</tr>
</tbody>
</table>

Comparison of displacements is presented in Figure 9.

Fig. 8. Calculation scheme for the single-pylon bridge

Maximum deflections of the girder are presented in Table 1.

4. Conclusion

The discrete method gives smallest deflection of the girder. Smallest displacement from traffic load $w_t = -0.208$ m come from non-linear scheme when cross-section of the cable is 45800 mm$^2$. Displacement from the traffic load in case of discrete scheme $w_t = 0.205$ mm is practically similar as displacement from non-linear scheme. Displacements from linear scheme are noticeably larger than displacements from non-linear or discrete scheme.

In case of self-anchoring scheme, displacement from load $p_1 = 20$ kN/m was $w_{p1} = 0.188$ m, displacement from load $p = p_1 + p_o = 50$ kN/m was $w_p = 0.395$ m and displacement from traffic load only was $w_t = 0.395 - 0.188 = 0.207$ m. This is practically same as displacements from bridge, where cables are anchored to the abutments.
sis. If discrete loadings are predicted for the suspension bridge, then the discrete calculation model should be used for the calculation of the bridge. The continual model fits well in situations where the influence of discrete loadings is insignificant. The continual method also has the advantage of good and quick convergence over the discrete model.

References


KABAMOJO TILTO SU VIENU PILONU DISKREČIOJI ANALIZĖ

M. Kiisa, J. Idnurm, S. Idnurm


Reikšminiai žodžiai: konstrukcijos, standumo sijos, kabamasis lynas, didelių tarpatramių konstrukcijos, diskrečioji analizė, geometrinis netiesiškumas.

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