DISCRETE MELLIN CONVOLUTION WITH DILATION AND ITS APPLICATIONS

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ABSTRACT

The analysis of the discrete Mellin convolution is given. A generalization of results from [4,5] is presented. Some applications illustrate the efficiency of proposed methods.

1. MAIN RESULTS

Let denote by $l_{\nu,p}$ — the Banach space of sequences $a(n)$, such that $a(n)n^{\nu-1/p} \in l_p$, $\nu \in \mathbb{R}$, $1 \leq p \leq \infty$, with norm

$$\|a(n)\|_{l_{\nu,p}} = \|a(n)n^{\nu-1/p}\|_p;$$

$l_{\nu,p}(x)$ — the Banach space of functional sequences $a(n,x), x \in (0, \infty)$, such that $a_n \in l_{\nu,p}$, where $a_n = \text{ess sup}_x |a(n,x)|$, with norm

$$\|a(n,x)\|_{l_{\nu,p}(x)} = \|a_n\|_{l_{\nu,p}};$$

$L_{\nu,p} —$ the Banach space of functions $f(x), x \in (0, \infty)$, such that $f(x)x^{\nu-1/p} \in L_p$, $\nu \in \mathbb{R}$, $1 \leq p \leq \infty$, with norm

$$\|f(x)\|_{L_{\nu,p}} = \|f(x)x^{\nu-1/p}\|_L.$$

DEFINITION 1. Let

$$h_\tau(n,x) = \sum_{kn = m} a(k,x)b(m,kx) = \sum_{k|n} a(n/k,x)b(k,(n/k)x), n \geq 1, \quad (1)$$
here \( \tau \neq 0 \), \( k|n \) means that \( k \) is divisor of \( n \). Sequence \( h_{\tau} (n, x) = (a * b)_{\tau} (n, x) \) is said to be discrete Mellin convolution of functional sequences \( a(n, x) \) and \( b(n, x) \) with \( \tau \)-degree dilation (\( DMC_{\tau} \)).

Under the fixed \( a(n, x) \) and \( \tau \) the \( DMC_{\tau} \) is a linear operator mapping sequence \( b(n, x) \) into \( h_{\tau} (n, x) \).

**Lemma 1.** Let \( a(n, x) \in l_{p+1/\alpha, 1}(x) \). Then the \( DMC_{\tau} \) is a bounded operator in \( l_{p, \nu}(x), \nu \in \mathbb{R}, 1 \leq p \leq \infty \), and

\[
\| (a * b)_{\tau} (n, x) \|_{l_{p, \nu}(x)} \leq \| a(n, x) \|_{l_{p+1/\alpha, 1}(x)} \| b(n, x) \|_{l_{p, \nu}(x)}.\]

**Proof.** It is evident that

\[
h_{\tau} (n, x) = \sum_{k \mid n} a(n/k, x)b(k, (n/k)^{\tau} x) = \sum_{k \mid n} b(n/k, k^\tau x)a(k, x).
\]

The convolution turns into

\[
h_{\tau} (n, x) = \sum_{k=1}^{\infty} b_{nk}(x, \tau)a(k, x),
\]

where \( B(x, \tau) = \{ b_{nk}(x, \tau) \} \) is a \( DMC_{\tau} \)-matrix. For \( p = \infty \)

\[
\| h_{\tau} (n, x) \|_{l_{p, \nu}(x)} = \sup_{n} \| b_{nk} \| = \sup_{n} \left( \nu \text{ess} \sup_{x} \left( \sum_{k=1}^{\infty} b_{nk}(x, \tau)a(k, x) \right) \right)
\]

\[
\leq \sup_{n} \left( \left( \frac{n}{k} \right)^{\nu} b_{nk} \right) \sum_{k=1}^{\infty} k^{\nu}a_{k} = \| a(n, x) \|_{l_{p+1/\alpha, 1}(x)} \| b(n, x) \|_{l_{p, \infty}(x)}.
\]

When \( 1 \leq p < \infty \) the result follows from the generalised Minkowsky inequality (see [1]):

\[
\| h_{\tau} (n, x) \|_{l_{p, \nu}(x)} = \left( \sum_{n=1}^{\infty} \left( \nu^{p-1/\alpha} \text{ess} \sup_{x} \left( \sum_{k=1}^{\infty} b_{nk}(x, \tau)a(k, x) \right) \right)^{p} \right)^{1/p}
\]

\[
\leq \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} a_{k}^{p} b_{nk}^{p} n^{\nu p-1} \right)^{1/p} = \| a(n, x) \|_{l_{p+1/\alpha, 1}(x)} \| b(n, x) \|_{l_{p, \nu}(x)}.
\]

\[\square\]

**Lemma 2.** Discrete Mellin convolution (1) is associative.
Proof. We have the following equalities

\[(a \ast b) \ast c)_r(n, x) = \sum_{km=n} (a \ast b)_r(k, x)c(m, k^\tau x)\]

\[= \sum_{km=n} \left( \sum_{tl=k} a(s, x)b(t, s^\tau x) \right) c(m, k^\tau x) = \sum_{sl=m=n} a(s, x)b(t, s^\tau x)c(m, s^\tau t^\tau x)\]

\[= \sum_{sl=m=n} a(s, x) ( \sum_{tm=w} b(t, s^\tau x)c(m, s^\tau t^\tau x) ) = \sum_{nw=m=n} a(s, x)(b \ast c)_r(w, s^\tau x)\]

\[= (a \ast (b \ast c)_r)(n, x).\]

 Definition 2. A sequence \(a^{-1}(n, x)\) is said to be reciprocal to \(a(n, x)\) with respect to the DMC, if almost everywhere on \((0, \infty)\)

\[(a \ast a^{-1})_r(n, x) = (a^{-1} \ast a)_r(n, x) = \delta_n = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}\]

Lemma 3. If \(\mathfrak{a}_{\text{inf}} = \text{essinf}_{x} |a(1, x)| > 0\), then the reciprocal sequence may be expressed by the recursion relation

\[a^{-1}(1, x) = \frac{1}{a(1, x)}, \quad (2)\]

\[a^{-1}(n, x) = - \frac{1}{a(1, n^\tau x)} \sum_{km=n} a^{-1}(k, x)a(m, k^\tau x)\quad \text{for } k < n\]

\[= - \frac{1}{a(1, x)} \sum_{km=n} a(k, x)a^{-1}(m, k^\tau x), \quad n > 1, \quad (3)\]

or in the explicit form

\[a^{-1}(n, x) = \frac{1}{a(1, n^\tau x)}\]

\[\sum_{\beta \in A_n} (-1)^{\|\beta\|} \sum_{i_{\beta}} \frac{a(i_1, x) a(i_2, i_1^\tau x) \cdots a(i_{\|\beta\|}, i_{\|\beta\|-1}^\tau x)}{a(1, i_1^\tau x) a(1, i_2^\tau x) \cdots a(1, i_{\|\beta\|-1}^\tau x)}, \quad (4)\]
where \( A_n = \left( \beta = (\beta_2, \beta_3, ..., \beta_n), \beta_k = 0, 1, 2, ..., \prod_{k=2}^{n} k^{\beta_k} = n, n \geq 1 \right), i(\beta) \) is a set of permutations of naturals corresponding to \( \beta_k \neq 0 \) (number \( k \) is taken \( \beta_k \) times), \( |\beta| = \sum_{k=1}^{n} \beta_k \). As soon as \( a(n, x) \equiv a(n) \) the last formula becomes

\[
a^{-1}(n) = \frac{1}{a(1)} \sum_{\beta \in A_n} \frac{(-1)^{|\beta|}}{\beta_2! \beta_3! \cdots \beta_n!} \frac{(a(2))^{\beta_2}}{a(1)} \frac{(a(3))^{\beta_3}}{a(1)} \cdots \frac{(a(n))^{\beta_n}}{a(1)}. \]

Proof. Formulae (2) and (3) directly follow from the Definition 1. Formula (4) will be proved by induction. For \( n = 1 \) (4) gives (2). Suppose (4) be true when \( n < k \). Then from the formula (3) and from the induction hypothesis for \( n = k \) we obtain

\[
a^{-1}(k, x) = -\frac{1}{a(1, k^2 x)} \sum_{s < k} a^{-1}(s, x) a(t, s^t x)
\]

\[
= -\sum_{s < k} \sum_{st = k} \frac{(-1)^{|\beta|}}{a(1, k^2 x)} a(i_{(\beta)}) \frac{a(i_{1}, x)}{a(1, x)} \frac{a(i_{2}, x)}{a(1, x)} \cdots \frac{a(i_{|\beta|}, x)}{a(1, x)} \frac{a(t, s^t x)}{a(1, s^t x)}
\]

\[
= \frac{1}{a(1, k^2 x)} \sum_{\alpha \in A_n} \frac{(-1)^{|\alpha|}}{a(1, k^2 x)} a(j_{(\alpha)}) \frac{a(j_{1}, x)}{a(1, x)} \frac{a(j_{2}, x)}{a(1, x)} \cdots \frac{a(j_{|\alpha|}, x)}{a(1, x)} \frac{a(t, s^t x)}{a(1, s^t x)}.
\]

since \( A_k = \{\alpha = (\beta_2, ..., \beta_k + 1, ..., \beta_k), \beta_k \in A_k, st = k\} \). Thus statement is proved for arbitrary \( n \). \( \square \)

**Theorem 4.** The existence of \( m \) sequences \( a_\mu(n, x) \) from \( l_{v, 1}(x) \), \( \mu = 1, ..., m \), such that

1) \( (a_1 * a_2 * ... * a_m)_x (n, x) = a(n, x); \)
2) \( |a_\mu(n, x)| \in l_{v, 1}(x) \) \( \leq \text{essinf}_x |a_\mu(1, x)| + \text{esssup}_x |a_\mu(1, x)| \)

\( = a_{\mu, \text{inf}} + a_{\mu, 1}, \mu = 1, ..., m \)

is sufficient for \( a^{-1}(n, x) \) belongs to \( l_{v, 1}(x) \).

Proof. It follows from 2) that the reciprocal sequence \( b^{-1}(n, x) \) belongs to \( l_{v, 1}(x) \) for any sequence \( b(n, x) \). In fact

\[
b^{-1}_n = \text{esssup}_x |b^{-1}(n, x)| \leq \sum_{\beta \in A_n} \frac{1}{|\beta|!} \frac{|\beta|!}{\beta_2! \beta_3! \cdots \beta_n!} b_\beta a_\beta_2 ... b_\beta_n.
\]
where \( b_{\inf} = \text{ess inf}_{x} |b(1, x)| \). Then

\[
\sum_{n=1}^{\infty} n^{\nu-1} b_{\inf}^{-1} \leq \sum_{n=1}^{\infty} \sum_{\beta \in A_{n}} \frac{1}{b_{\inf}^{\beta_{1}+1} \beta_{2}! \cdots \beta_{n}!} b_{\inf}^{\nu-1} b_{\inf} \beta \sum_{k=0}^{s} \frac{1}{k!} b_{\inf}^{\nu-1} b_{\inf} \beta_{k} \cdot \beta \sum_{k=0}^{s} \frac{1}{k!} b_{\inf}^{\nu-1} b_{\inf} \beta_{k}.
\]

Thus each sequence \( a_{\nu-1}^{-1} (n, x) \in l_{\nu, 1} (x) \). We deduce from Lemmas 1, 2 that \( a^{-1} (n, x) \in l_{\nu, 1} (x) \). □

The conditions of theorem 4 are best possible, because there are sequences for which these conditions are necessary and sufficient. For example, \( a(n, x) = (1, \alpha, 0, \ldots, 0, \ldots) \).

Examine the operator

\[
(M_{a, \tau} f) (x) = \sum_{n=1}^{\infty} a(n, x) f(n^{\tau} x), \tau \neq 0, x \in (0, \infty).
\]

**Lemma 5.** If \( a(n, x) \in l_{1-\tau, 1} (x) \), then \( M_{a, \tau} \) (5) is bounded operator in \( L_{\nu, p}, \nu \in \mathbb{R}, 1 \leq p \leq \infty \).

Proof of the lemma follows from the generalised Minkowsky inequality.

**Theorem 6.** Suppose \( a(n, x), b(n, x) \in l_{1-\tau, 1} (x) \). For arbitrary function \( f(x) \in L_{\nu, p}, \nu \in \mathbb{R}, 1 \leq p \leq \infty \) it is true that

\[
(M_{a, \tau} M_{b, \tau} f) = M_{b, \tau} f,
\]

where \( M_{b, \tau} \) is the operator (5) corresponding to \( h_{\nu} (n, x) = (a * b)_{\nu} (n, x) \).

**Proof.**

For any function \( f(x) \in L_{\nu, p} \)

\[
(M_{a, \tau} (M_{b, \tau} f)) (x) = \sum_{n=1}^{\infty} a(n, x) (M_{b, \tau} f) (n^{\tau} x)
\]
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\[ a(n, x) \sum_{k=1}^{\infty} b(k, n^\tau x) f(n^\tau k^\tau x) = \lfloor nk = m \rfloor \]

\[ = \sum_{m=1}^{\infty} \left( \sum_{n,k=m} a(n, x)b(k, n^\tau x) \right) f(m^\tau x) = (M_{k,\tau} f)(x). \]

Rearrangement of summands is possible due to Lemmas 1, 5. □

**Corollary 1.** Under conditions of the Theorem 4 the following formula is true for arbitrary function \( f(x) \in L_{\nu,p}, \nu \in \mathbb{R}, 1 \leq p \leq \infty \)

\[ M_{a,\tau}^{-1} f = M_{a^{-1},\tau} f. \] (6)

Here \( a^{-1}(n, x) \) is reciprocal sequence to \( a(n, x) \) with respect to the DMC\(_\tau\).

Proof of this corollary follows from Theorems 4, 6.

Let us consider the following integral equations of the Mellin convolution type on \((0, \infty)\):

\[ \int_0^\infty k(x, t) f(t) \frac{dt}{t} = g(x), \] (7)

\[ \int_0^\infty k(t, x) f(t) \frac{dt}{t} = g(x), \] (8)

\[ \int_0^\infty k(x, xt) f(t) dt = g(x), \] (9)

\[ \int_0^\infty k(t, xt) f(t) dt = g(x), \] (10)

Such equations are well-known when \( k(u, w) \equiv m(w) \) is a hypergeometric type function, see [1], [2]. We solve (7)-(10) with the kernels of another special type:

\[ k(u, w) = \sum_{n=1}^{\infty} a(n, u)m(n^\tau w), \tau \neq 0, u, w \in (0, \infty). \] (11)

Here we suppose that solutions of (7)-(10) with \( k(u, w) \equiv m(w) \) are known.

**Lemma 7.** Let \( k(w) = \text{esssup} |k(u, w)| \in L_{\nu,1}. \) The operators from the left parts of (7), (8) [(9), (10)] are bounded ones from \( L_{\nu,p} [L_{1-\nu,p}], \nu \in \mathbb{R}, 1 \leq p \leq \infty \) in \( L_{\nu,p}. \)
Lemma 8. If \( a(n, x) \in \ell_{1-\tau n, 1}(x), m(x) \in L_{\nu, 1} \), then (11) is satisfied assumption of Lemma 7.

Proof of the Lemmas 7, 8, immediately follows from the generalised Minkovskiy inequality. In this case rearrangement of summing and integrating is possible due to the analogue of the Fubini theorem [3].

Using the obtained results we can express equations (7)-(10) with kernel (11) in the form

\[
(M_{a, \tau} Q f)(x) = \sum_{n=1}^{\infty} a(n, x) \int_{0}^{\infty} m\left(\frac{n^{-1} x}{t}\right) f(t) \frac{dt}{t} = g(x),
\]

(12)

\[
(Q M_{b, \tau} f)(x) = \int_{0}^{\infty} m\left(\frac{x}{t}\right) \left(\sum_{n=1}^{\infty} b(n, t) f(n^{-1} t)\right) \frac{dt}{t} = g(x),
\]

(13)

\[
(M_{a, \tau} K f)(x) = \sum_{n=1}^{\infty} a(n, x) \int_{0}^{\infty} m(n^{-1} x t) f(t) dt = g(x),
\]

(14)

\[
(K M_{c, \tau} f)(x) = \int_{0}^{\infty} m(x t) \left(\sum_{n=1}^{\infty} c(n, t) f(n^{-1} t)\right) dt = g(x),
\]

(15)

where

\[
b(n, x) = a(n, n^{-1} x), \quad c(n, x) = a(n, n^{-1} x)n^{-\tau},
\]

All series in (12)-(15) converge in mean under the Lemmas’ 7, 8 conditions. With Theorems 4, 6 and formula (6) we obtain the solutions of equations in the following form:

\[
f(x) = (Q^{-1} M_{a^{-1}, \tau} g)(x),
\]

(16)

\[
f(x) = (M_{b^{-1}, \tau} Q^{-1} g)(x),
\]

(17)

\[
f(x) = (K^{-1} M_{a^{-1}, \tau} g)(x),
\]

(18)

\[
f(x) = (M_{c^{-1}, \tau} K^{-1} g)(x).
\]

(19)

This paper generalize the results of [4], [5].
REFERENCES


