ON GENERAL REPRESENTATION OF THE MEROMORPHIC SOLUTIONS OF HIGHER ANALOGUES OF THE SECOND PAINLEVE EQUATION

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One of the important questions of the nonlinear ordinary differential equations theory is representation of the meromorphic solutions as the ratio of the entire functions similarly to the Weierstrass function $\rho(z)$, which is a solution of an equation $\rho'' = 4\rho^3 + 2\rho + g_3$, and it has representation through the entire function $\sigma(z)$

$$\rho(z) = \frac{\sigma'' - \sigma\sigma''}{\sigma^2} = -\zeta'(z), \quad \zeta = (\ln(\sigma(z)))'$$

We shall consider a reduction of the higher - Korteweg-de Vries equations

$$(2m - 1)u_t = X_m u,$$  \hspace{1cm} (1)

where $X_1 u = Du = D\frac{\partial H}{\partial u}$, $D = \frac{\partial}{\partial x}$, $X_m u = (2u + 2DuD^{-1} - D^2)X_{m-1} u = D\frac{\partial H}{\partial u}$, $m = 2, 3, \ldots$

applying the formulas

$$z = xt^{-\frac{m-1}{m}}, \quad u(x, t) = t^{-\frac{m}{m-1}} \left( \frac{dw}{dz} + w^2 \right)$$ \hspace{1cm} (2)

to the ordinary differential equation

$$D^{-1} S^{m-1}_{w'} + zw + \delta = 0,$$ \hspace{1cm} (mP_2)
where $S_w = 4w^2 + 4w' D_w^{-1} - D^2$, $D = d/dz$.

The order of equation $(m P_2)$ is $2m - 2$. For $m = 2$ equations (1) and $(m P_2)$ become the Korteweg - de Vries equation and the second Painlevé equation ($P_2$) correspondingly. Let us call equation $(m P_2)$ the higher analogue of the second Painleve equation similarly to equation (1). For $m = 3$ we have

$$w^{(4)} = 10w^2w'' + 10ww'^2 - 6w^5 - zw - \delta. \quad (3 P_2)$$

It is well known, that solutions of the Painlevé equations are, in a general case, meromorphic functions, which we may not state definitely about the higher analogues of equation $P_2$. In paper [1] representation of the Painlevé equations’ solutions as the ratio of the entire functions was given and one-to-one correspondence between the solutions of these equations and the systems constructed for them was established.

In the present paper we offer representation of the meromorphic solutions of equations $(m P_2)$ as the ratio

$$w(z) = \frac{v(z)}{u(z)} \quad (3)$$

of entire functions $v(z), u(z)$. We establish one-to-one correspondence between the meromorphic solutions of the equation $(m P_2)$ and entire solutions of the constructed system.

For the equation $(m P_2)$ let’s assume following [2, 3, 1]

$$u(z) = \exp\left(-\int_{z_0}^{z} dz \int_{z_0}^{z} w^2 dz\right), \quad (4)$$

where the path of integration does not pass through the singularities of the function $w(z)$. In the neighborhood of the movable pole $z = \alpha$ the meromorphic solution $w(z)$ has expansion [4]

$$w = a_{-1}(z - \alpha)^{-1} + a_1(z - \alpha) + \varphi_1(z, \alpha), \quad (5)$$

where $a_{-1}$ takes any of the values $\pm 1, \pm 2, \ldots \pm (m - 1)$, and $\varphi_1(z, \alpha)$ is an analytic function in the neighborhood of $\alpha$. Then from (4) it follows that $u(z)$ is an entire function for any meromorphic solution of the equation $(m P_2)$, and at pole $\alpha$ of the solution $w(z)$ the function $u(z)$ has zero of order $a_{-1}^2$. This fact is easily established by substitution (5) into the right part of (4). Hence, if we will define function $v(z)$ as $v(z) = w(z)u(z)$, then the function $v(z)$ is also an entire one for any meromorphic solution of the equation $(m P_2)$. The system for finding of the entire functions $u(z), v(z)$ turns out by differentiation (4) by virtue of (3) and substitution (3) into the equation $(m P_2)$. It has the form

$$uu'' - u^2 = v^2, \quad D^{-1} S_{uu^{-1}}(u' u^{-1} - vu' u^{-2}) + zvu^{-1} + \delta = 0. \quad (6)$$
On general representation of meromorphic solutions

For example, at \( m = 3 \) for the equation \((3P_2)\) we have

\[
\begin{align*}
u u'' - u^2 &= v^2, \\
v^{(4)} u^4 - 4v'' u v u^3 + 6v'' u^2 u^2 - 2v'' v^2 u^2 - 4v' v^3 u \\
+ 4v^2 v' u' + vu'' - 2v^3 u' + v^5 + z u^4 + \delta u^5 &= 0.
\end{align*}
\]

Let us consider system (6).

**Lemma 1.** The system (6) has the solution

\[
v = 0, \quad u = \exp(az + b) \tag{7}
\]

for any \( a, b \in C, \ \delta = 0. \)

The choice of the solution of system (6) is defined by the initial conditions

\[
u(z_0) = u_0, \ \nu'(z_0) = u'_0, \ \nu(z_0) = v_0, \ \nu'(z_0) = v'_0, \ldots, \ v^{(2m-3)}(z_0) = v^{(2m-3)}_0, \tag{8}
\]

where \( z_0, \ u_0, \ u'_0, \ v_0, \ v'_0, \ldots, \ v^{(2m-3)}_0 \in C. \) Thus, if \( u(z_0) = 0 \), then the initial conditions (8) is singular.

**Lemma 2.** If \((v, u)\) is a solution of system (6), then

\[
(\delta, \bar{u}) = (\lambda(z) v, \lambda(z) u), \quad \lambda(z) \neq 0 \tag{9}
\]

is a solution of system (6) if and only if \( \lambda = e^{az+b}, \ a, b \in C. \)

The validity of lemmas 1 and 2 are confirmed by direct substitution (7) and (9) into system (6).

**Lemma 3.** Any solution of system (6), corresponding to the meromorphic solution of an equation \((mP_2)\) and satisfying to the initial conditions

\[
u(z_0) = 1, \ \nu'(z_0) = 0, \ \nu(z_0) = v_0, \ \nu'(z_0) = v'_0, \ldots, \ v^{(2m-3)}(z_0) = v^{(2m-3)}_0, \tag{10}
\]

is an entire one.

**Proof.** The initial conditions (10) are not singular. Let \( w(z) \) be a meromorphic solution of an equation \((mP_2)\) with the initial conditions \( w(z_0) = w_0, w'(z_0) = w'_0, \ldots, w^{(2m-3)}(z_0) = w^{(2m-3)}_0. \) We shall consider the functions

\[
u_1(z) = \exp(- \int_{z_0}^{z} dz' \int_{z_0}^{z} u^2 dz'), \quad v_1(z) = w(z) u_1(z). \tag{11}
\]

From construction functions \( u_1(z), v_1(z) \) are the entire ones and satisfy to system (6). By virtue of uniqueness the statement will be proved if we take \( w_0 = v_0, w'_0 = v'_0, \ldots, w^{(2m-3)}_0 = v^{(2m-3)}_0. \)
Lemma 4. Any solution of system (6), corresponding to the meromorphic solution of equation \((m,P_2)\) and satisfying to the initial conditions

\[
\begin{align*}
  u(z_0) &= u_0 \neq 0, & u'(z_0) &= u'_0, & v(z_0) &= v_0, & v'(z_0) &= v'_0, & \\
  v^{(2m-3)}(z_0) &= v_0^{(2m-3)},
\end{align*}
\]

is an entire one.

Proof. We shall take the solution \((\tilde{v}, \tilde{u})\) of system (6) with the initial conditions

\[
\tilde{u}(z_0) = 1, \quad \tilde{u}'(z_0) = 0, \quad \tilde{v}(z_0) = \tilde{v}_0, \quad \tilde{v}'(z_0) = \tilde{v}_0', \ldots, \quad \tilde{v}^{(2m-3)}(z_0) = \tilde{v}_0^{(2m-3)}.
\]

By virtue of lemma 3 this solution is an entire one, and by virtue of lemma 2 the functions \((v, u) = (\tilde{v} \exp(az + b), \tilde{u} \exp(az + b))\) will also be an entire solution of system (6). The proof of lemma 4 follows from the choice \(a, b\) and \(\tilde{v}_0, \tilde{v}_0', \ldots, \tilde{v}_0^{(2m-3)}\) from a condition

\[
a = \frac{v_0'}{u_0}, \quad b = \ln(u_0) - \frac{v_0'^2}{u_0'^2}, \quad \tilde{v}_0 = \frac{v_0}{u_0}, \ldots, \quad \tilde{v}_0^{(2m-3)} = \left(\frac{v}{u}\right)^{(2m-3)}(z_0),
\]

where \(u''(z_0) = u_0''/u_0\).

Theorem 1. All solutions of system (6), corresponding to the meromorphic solutions of equation \((m,P_2)\), are entire functions.

Proof. If \(u \equiv 0\), then \(v \equiv 0\) and this solution is an entire one. Let \(u \neq 0\). Then there exists domain \(D\) where function \(u(z) \neq 0\) and it is an analytic one. Let \(z_0 \in D\). Then by virtue of lemma 4 we have the required statement.

Theorem 2. Let \((v, u)\) be an arbitrary entire non-zero solution of system (6) for some fixed value of parameter \(\delta\), which is different from the solutions (7). Then a ratio \(v(z)/u(z)\) represents meromorphic solution of equation \((m,P_2)\) for the same value of parameter \(\delta\).

Proof. Let \((v, u)\) be an entire non-zero solution of system (6). Then there exists such \(z_0\), that \(u(z_0) = u_0 \neq 0\). Let us take the meromorphic solution \(w(z)\) of equation \((m,P_2)\) with the initial conditions \(w(z_0) = v_0/u_0, \quad w'(z_0) = v'_0/u_0 - v_0u'_0/u_0^2, \ldots, \quad w^{(2m-3)}(z_0) = (v/u)^{(2m-3)}(z_0)\), where we assume \(u''(z_0) = u_0''/u_0\). We shall construct a solution of system (6) applying the formula

\[
u_1(z) = u_0 \exp((z - z_0)u_0' - \int_{z_0}^z dz \int_{z_0}^z w'(z)dz), \quad v_1(z) = u_1(z)w(z).
\]

It is not difficult to see that the solution \((v_1(z), u_1(z))\) satisfies to the same initial conditions, as \((v(z), u(z))\). In view of this by virtue of uniqueness
On general representation of meromorphic solutions

(v_1(z), u_1(z)) \equiv (v(z), u(z)) and from the second parity of (11) statement of the theorem follows.

**Theorem 3.** Any meromorphic solution \( w(z) \) of equation \((mP_2)\) is represented in the form \( w(z) = v(z)/u(z) \), where \((v(z), u(z))\) is the corresponding entire solution of system (6), determined up to the factor \( \exp(az + b) \).

**Proof.** If \( w(z) \equiv 0 \) at \( \delta = 0 \), we shall take the entire solution of system (6) \((v(z), u(z)) = (0, \exp(az + b))\). Let \( w(z) \neq 0 \). Then there exists such \( z_0 \), that \( w(z_0) \neq 0, w(z_0) \neq \infty \). Let us put

\[
   u(z) = \exp\left(-\int_{z_0}^{z} dz \int_{z_0}^{z} w^2 dz\right), \quad v(z) = w(z)u(z).
\]

These functions are entire solutions of system (6), and \( w(z) = v(z)/u(z) \). But by virtue of (9) \( w(z) \) is also expressed through the solution

\[
   (\tilde{v}(z), \tilde{u}(z)) = (v(z) \exp(az + b), u(z) \exp(az + b)).
\]

The theorems 2 and 3 establish one-to-one correspondence between the meromorphic solutions of equation \((mP_2)\) and the entire solutions of system (6).

**REFERENCES**


