THE NONLINEAR CONVOLUTIONAL EQUATIONS SOLVABLE IN CLOSED FORM

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1. SECTION 1

Let $C: H \leftrightarrow W$ is bijective linear operator, operating from algebra $W$ in linear space $H$. We define convolution with weight $\gamma \in W$ by equation [1,2]

$$\varphi \ast \psi = C^{-1} [\gamma \cdot (C\varphi) \cdot (C\psi)], \varphi, \psi \text{ and } \varphi \ast \psi \in H$$

Then

$$\varphi \ast \psi = \psi \ast \varphi \text{ and } (\varphi \ast \psi) \ast \chi = \varphi \ast (\psi \ast \chi), \forall \varphi, \psi, \chi \in H$$

$$(\varphi \ast \psi) \ast \chi = C^{-1} \{\gamma \cdot [C(\varphi \ast \psi) \cdot (C\chi)] = C^{-1} [\gamma^2 \cdot C(\varphi) \cdot C(\psi) \cdot (C\chi)]$$

Then from here the possibility of definition of “power” of convolution follows [3,4]

$$(\varphi)^n = \underbrace{\varphi \ast \varphi \ast \ldots \ast \varphi}_{n} = C^{-1} [\gamma^{n-1} \cdot C(\varphi)^n], n \in N.$$

Particular cases are:

$$(\varphi)^1 = \varphi, \ (\varphi)^n \ast (\varphi)^m = (\varphi)^{n+m} \forall n, m \in N.$$

We shall suppose further, that the algebra $H$ is commutative normalized ring [5] concerning of norm $||| \cdot |||$ such, that

$$||| \varphi \ast \psi ||| \leq ||| \varphi ||| \cdot ||| \psi |||, \forall \varphi, \psi \in H \text{ and, particularly, } |||(\varphi)^n||| \leq |||\varphi|||^n \quad (1)$$

At last, we shall suppose, that for element $\varphi \in H$ exists finite or infinite limit

$$\lim_{n \to \infty} [\gamma^{n-1} \cdot (C\varphi)^n] = \varphi_{\infty}$$

where from $\lim_{n \to \infty} (\varphi)^n = C^{-1}(\varphi_{\infty})$ or belong to $H$, or this limit is infinite.
2. SECTION 2

Assuming, that $\gamma^{-1} \in W$, power series has form (3)

$$y = a(x) = \sum_{n=1}^{\infty} a_n x^n, \quad a_1 = 1, \quad |x| < r_a.$$  (3)

and we compare “convolutional” series (compare [4]) to power series (3)

$$\sum_{n=1}^{\infty} a_n (\varphi)^n = \sum_{n=1}^{\infty} a_n C^{-1} \left[ \gamma^{n-1} \cdot C(\varphi)^n \right] = C^{-1} \left[ \sum_{n=1}^{\infty} a_n \gamma^{n-1} \cdot C(\varphi)^n \right] = C^{-1} \left\{ \gamma^{-1} \cdot \sum_{n=1}^{\infty} a_n [\gamma^n \cdot C(\varphi)^n] \right\},$$  (4)

which, obviously, make sense, if $||\varphi|| \leq r_a$.

Let the series

$$x = \sum_{k=1}^{\infty} b_k y^k, \quad b_1 = 1, \quad |y| < r_b$$  (5)

convert series (3):

$$b(a(x)) = x, \quad |x| < s_a < r_a \quad a(b(y)) = y, \quad |y| < s_b < r_b,$$  (6)

then the factors $b_k$ of series (5) determining (see p. 755 in [6]) by formulae:

$$b_m = \sum_{a_2 + \cdots + (m-1) a_m = m-1} \frac{(m-1 + a_2 + \cdots + a_m)!}{m! \alpha_2! \cdots \alpha_m!} \cdot (-a_2)^{a_2} \cdots (-a_m)^{a_m},$$  (7)

where $\alpha_j \in \mathbb{N}$.

In particular, if series (3) is polynomial of power $p$, then instead (7) we have:

$$b_m = \sum_{a_2 + \cdots + (p-1) a_p = m-1} \frac{(m-1 + a_2 + \cdots + a_p)!}{m! \alpha_2! \cdots \alpha_p!} \cdot (-a_2)^{a_2} \cdots (-a_p)^{a_p}.$$  (8)
3. SECTION 3

We consider convolutional equation

\[ a(\varphi) = \sum_{n=1}^{\infty} a_n(\varphi)^n = \psi, \quad (9) \]

in which \( \psi \) is known, and \( \varphi \) is required element from \( H \). In general case this equation have infinite order, but it may be polynomial equation of power \( p \geq 2 \).

We allows, that function \( a(x) \) is invertible with help series (5) and, hence, the equality (6) and (7) (or (8)) take place. We notice, that function \( a(x) \), is strictly monotonic on interval \( \{ |x| < s_a < r_b \} \), is obviously invertible.

Taking into account (4) we find that \( b(\gamma \cdot (C\psi)) = \gamma \cdot (C\varphi) \) or

\[ \varphi = C^{-1} \left[ \gamma^{-1} \cdot b(\gamma \cdot (C\psi)) \right] = \sum_{n=1}^{\infty} b_n(\psi)^n. \quad (10) \]

From here, if \( \|\psi\| < s_b \), then equality (9) has the solution \( \varphi \in H \), determined by formula (10) \( \|\varphi\| < s_a \) and by virtue of (1).

It is easy to see, that convolutional equation

\[ \sum_{n=1}^{\infty} b_n(\varphi)^n = \psi, \quad \psi \in H, \quad \|\psi\| < s_b \]

has the solution

\[ \varphi = \sum_{n=1}^{\infty} a_n(\psi)^n \in H, \quad \|\varphi\| < s_a. \]

4. SECTION 4

We made two remarks:

a) Departing from decomposition of kind [7]

\[ \sin \lambda x = 2 \sum_{n=1}^{\infty} (-1)^{n+1} J_{2n-1}(\lambda) T_{2n-1}(x) = \]

\[ \frac{2}{\lambda} \sum_{n=1}^{\infty} (-1)^n J_{2n}(\lambda) U_{2n-1}(x), \quad |x| < 1, \quad 0 < \lambda < \pi \]

naturally series (3) to replace on more general series

\[ \sum_{n=1}^{\infty} b_n Q_{2n-1}(x), \quad Q_{2n-1}(x) = \sum_{s=1}^{n} q_{n,s} x^{2s-1}. \]

b) Probably, for the first time particular nonlinear equation in convolutions with \( p=2 \) was investigated in article [8]. One of variants of research of polynomial equations with convolutions is offered in [4].
REFERENCES


