THE EVOLUTION OF THE PANMIGRATION POPULATION
TAKING INTO ACCOUNT DESTRUCTION OF THE FOETUS

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This article is devoted to the solvability of two models of nonmigrating nonlimited populations proposed in [1]. These models take into account age and sex of individuals, the panmictic mating and the destruction of the foetus (abortion). One of these models ignores but other one deals with the females restoration periods after abortions and deliveries. Existence and uniqueness theorems are proved as well as the estimate of solution is obtained.

1. The model ignoring females restoration intervals. We first put in remembrance the following notions used in [1]:

- $\tau_y, \tau_x$ and $\tau_z$ are ages of males, females and embryos, respectively, $t$ is time;
- $y(t,\tau_y), x(t,\tau_x)$ and $z(t,\tau_y,\tau_x,\tau_z)$ are densities of numbers of males, single and fecundated females, respectively;
- $p(t,\tau_y,\tau_x)$ is the females' fecundation rate;
- $v^y(t,\tau_y), v^x(t,\tau_x)$ and $v^z(t,\tau_y,\tau_x,\tau_z)$ are death rates of males, single and fecundated females, respectively;
- $\chi(t,\tau_y,\tau_x)$ is the abortions rate;
- $\sigma_z = [0,\kappa], 0 < \kappa < \infty$ is the females' gestation interval $\sigma_z = [0,\kappa], \sigma_x(\tau_z) = (\tau_1 + \tau_2 + \tau_3, \tau_2 + \tau_2), 0 < \tau_1 < \tau_2$;
- $\sigma_x(0), \sigma_y = (\tau_1,\tau_2)$, where $0 < \tau_1 < \tau_2 < \infty$, and $\sigma_x(\kappa)$ are the females' fecundation interval, the males' sexual activity interval and the females' reproductivity interval, respectively;
- $b^y(t,\tau_y,\tau_x)$ and $b^x(t,\tau_y,\tau_x)$ are the numbers of males and females, born from a female on a delivery;
- $[x(t,\tau_y)]$ is a jump of the function $x$ at the line $\tau_x = \tau_y$;
- $2^{-1/2}D^y, 2^{-1/2}D^x, 3^{-1/2}D^z$ reprezent directional derivatives along the positive direction of characteristics of operators $L^y = \partial/\partial t + \partial/\partial \tau_y, \quad L^x = \partial/\partial t + \partial/\partial \tau_x, \quad L^z = L^x + \partial/\partial \tau_z$ respectively;
- $\sigma = \sigma_y \times \sigma_x(\kappa), I = (0,\infty), I = [0,\infty), E^y = \{(t,\tau_y) \in I \times [0,\infty)\}$,
- $E^x = \left\{(t,\tau_x) \in I \times \bigcup_{i=1}^{4} \{t_i = t_{1x} + \kappa, t_2 = t_{2x} + \kappa, t_3 = t_{3x} + \kappa, t_4 = t_{4x} + \kappa, t_i > \kappa\} \right\},$
- $E^z = \{(t,\tau_y,\tau_x,\tau_z) \in I \times \sigma_y \times \sigma_x(\tau_z) \times \sigma_z\}$. 

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\[ \omega(\tau) = \sigma \times \Omega(\tau), \quad \Omega(\tau) = \begin{cases} [0, \tau - \tau_1], & \tau \in (\tau_1, \tau_2], \\ [\tau_3 - \tau_3, \tau], & \tau \in (\tau_3, \tau_4) \end{cases} \]

The inequality \( \tau_3 - \tau_1 > \kappa \) means the multiple deliveries. In this section the females organism restoration periods after abortions and deliveries will be ignored. The system (see [11])

\[ D^y y = -y v^y \quad \text{in} \quad E^y \]
\[ D^z z = -z d^z, \quad d^z = v^z + \chi \quad \text{in} \quad E^z \]
\[ D^x x = -x d^x + x_a + x_b \quad \text{in} \quad E^x \]
\[ d^x = v^x + \begin{cases} 0, & \tau \notin \sigma(0), \\ n^{-1} \int y p d\tau, & \tau \in \sigma(0), \end{cases} \]
\[ X_b = \begin{cases} 0, & \tau \notin \sigma(\kappa), \\ \int z_{\kappa} d\tau, & \tau \in \sigma(\kappa), \end{cases} \]
\[ X_a = \begin{cases} 0, & \tau \notin \sigma(\tau_1, \tau_4), \\ \omega(\tau), & \tau \in (\tau_1, \tau_4) \end{cases} \]

supplemented by the conditions

\[ x_{t=0} = x^0, \quad y_{t=0} = y^0, \quad z_{t=0} = z^0 \]
\[ y_{t=0} = \int b^y z_{\kappa} d\tau, \quad x_{t=0} = \int b^x z_{\kappa} d\tau \]
\[ z_{t=0} = n^{-1} x y p \]
\[ x_{\kappa} = 0, \quad i = 1, 4 \]

governs the evolution of the population. The non-negative demographic functions \( v^y, v^x, v^z, p, b^x, b^y, \chi \) and initial functions \( y^0, x^0, z^0 \) are assumed to be given. It is also assumed, that \( x^0, y^0, z^0 \) satisfy the reconcilability conditions, i.e. conditions (7) - (10) for \( t = 0 \). As it follows from the biological meaning, unknown functions \( y, x \) and \( z \) must be non-negative. We note, that \( (t, \tau), (t, \tau_y) \) and \( (t, \tau_y, \tau_x, \tau_z) \) are the arguments of functions \( d^x, X_b, X_a, v^y, \) and \( d^z \), respectively.

Now we consider the solvability problem of the model (1) - (10). Defining

\[ a = \sup_{(0, \tau_1]} x^0, \quad p^* = \sup_{E^p} p, \quad \chi^* = \sup_{E^\chi} \chi, \quad d^z = \inf_{E^x} d^x, \quad v^y = \inf_{E^y} v^y, \]
\[ v^x = \inf_{E^x} v^x, \quad b^* = \max_{i = x, y} \sup_{E^i} b d\tau, \quad q = \frac{b^*}{a} \int \sup_{E^y} z^0 d\tau \]

where \( \omega_1 = I \times \sigma y \times \sigma x(0), \quad \omega_2 = \sigma_x(\tau_2) \times \sigma_z \), we formulate the following statement.

**THEOREM 1.** Assume that:

1) \( \tau_3 - \tau_1 > \kappa \),
2) the nonnegative bounded functions $b^x, b^y$ are continuous in $t$ and piecewise continuous with respect to $\tau = (\tau_y, \tau_x)$ in $I \times \sigma$;

3) demographic functions $p, v^x, v^y, \nu^x, \nu^y, \chi$ and initial functions $x^0, y^0, z^0$, satisfying the reconcilable-conditions (7)–(10) for $t = 0$, are continuous and bounded;

4) the constants $\alpha, p^*, x^*, d^x, d^y, v^x, v^y, b^*, q$, defined by (11), are finite, positive and such that

$$b^*p^*\exp\left[-\kappa d^x\right] \leq q \leq 1, \quad q / b^* + x^* p^* \kappa \left(1 - \exp\left[-\kappa d^x\right]\right) \leq v^x, x^* p^* \leq d^y v^y \left[1 - \exp\left[-\kappa d^y\right]\right]$$

Then the problem (1)–(10) has a unique nonnegative solution such that:

1) $y, x$ and $z$ are continuous functions in $I \times I, I \times I$ and $I \times \sigma_y \times \sigma_x (\tau_z) \times \sigma_z$, respectively,

$$\max \left(\sup_t x(t, 0), \sup_t y(t, 0) \leq aq^{k+1}, \quad k \tau_4 < t \leq (k+1) \tau_4, \quad (k \tau_4 < t \leq (k+1) \tau_4) \right)$$

$$\left\{ \begin{array}{ll}
y_j(t) \leq y_0(t) \exp\left[-\nu_j^x\right], & 0 < t \leq \tau_y < \infty, \\
y_j(t) \leq aq^{k+1} \exp\left[-\nu_j^x\right], & k \tau_4 < t - \tau_y \leq (k+1) \tau_4, \quad \tau_y \in I,
\end{array} \right.$$  

$$\left\{ \begin{array}{ll}
x_j(t) \leq x_0(t) \exp\left[-\nu_j^x\right], & 0 < t \leq \tau_x \leq (0, \tau_2), \\
x_j(t) \leq aq^{k+1} \exp\left[-\nu_j^x\right], & k \tau_4 < t - \tau_x \leq (k+1) \tau_4, \quad \tau_x > \tau_4,
\end{array} \right.$$  

$$\left\{ \begin{array}{ll}
x_j(t) \leq aq^{k+1} \exp\left[-\nu_j^x\right], & k \tau_4 < t - \tau_x \leq (k+1) \tau_4, \quad \tau_x \in (0, \tau_2),
\end{array} \right.$$  

$k = 0, 1, 2, \ldots$

Proof. Let's denote: $\tau_0 = 0, \tau_5 = \infty, d^y = v^y, X = X_a + X_b,

$$\bar{x}(t) = x_{\tau_x} = 0, \bar{y}(t) = y_{\tau_y} = 0, \bar{z}(t, \tau_y, \tau_x) = z_{\tau_5} = 0.$$  


$$F_1(\gamma) = \gamma (r_\gamma^x) \exp\left[-\int_0^{\tau_x} d^{\gamma}(r_\gamma^x) d\gamma \right] + \int_0^{\tau_x} \exp\left[-\int_0^\tau d^{\gamma}(r_\gamma^x) d\tau \right] X(r_\gamma^x) d\gamma,$$  

$$F_2(\gamma, \mu) = \gamma (r_\mu^z) \exp\left[-\int_0^{\tau_x} d^{\gamma}(r_\mu^z) d\gamma \right] + \int_0^{\tau_x} \exp\left[-\int_0^\tau d^{\gamma}(r_\mu^z) d\tau \right] X(r_\mu^z) d\gamma,$$  

where $X(r_\gamma^x) = 0, X(r_\mu^z) = 0$ for $\gamma = y, z$ and $r_\gamma^x = (\eta, \eta + \tau_y - t), r_\mu^z = (\eta + t - \tau_y, \eta + \tau_x - \tau_z, \eta).$ If in (15) the letter $\gamma$ is not index, then it denotes the respective function. Let $\bar{x}, \bar{y}$ and $\bar{z}$ be the continuous functions and assume $d^x$ and $X$ are continuous functions except lines $\tau_x = \tau_i, i = 1, 2.$ Then from (1)–(3), (7), (15) we obtain the following integral representations.
\[
\begin{align}
\gamma &= \begin{cases} 
F_1(y), & y(h_2^o) = y(\tau_y - t), \quad 0 \leq t \leq \tau_y, \\
F_2(y,0), & y(h_2^o) = \bar{y}(\tau_y - t), \quad 0 \leq \tau_y < t,
\end{cases} \\
\bar{z} &= \begin{cases} 
F_1(z), & z(h_2^o) = z(\tau_y, \tau_x - t, \tau_z - t), \quad 0 \leq t \leq \tau_z \leq \kappa, \\
F_2(z,0), & z(h_2^o) = \bar{z}(\tau_x - \tau_z, \tau_y, \tau_x - \tau_z), \quad 0 \leq \tau_z < t,
\end{cases} \\
\bar{x} &= \begin{cases}
F_1(x), & x(h_2^o) = x(\tau_x - t), \quad 0 \leq t \leq \tau_x - \tau_i, \tau_x \in (\tau_i, \tau_{i+1}], \\
F_2(x, \tau_i), & t > \tau_x - \tau_i, \tau_x \in (\tau_i, \tau_{i+1}], x(h_2^o) = \bar{x}(t - \tau_x), \quad t > \tau_x \in [0, \tau_1],
\end{cases}
\end{align}
\]

where \( i = 1, 4. \)

The appearance of the index \( i \) is conditioned by the jumps of functions \( \alpha^x, X_a, X_b \).

Appending to (15)–(18) the formulas (4)–(6), (8), (9) we obtain the system of integral equations.

By using (17) we can write Eqs. (8), (5) and (6) in the form

\[
\gamma = \begin{cases} 
\bar{y} = \int_{[0, \kappa]} b^1 F_1(z) h_2^o \, d\sigma_y d\sigma_x, & \gamma = x, y \quad \text{in} \quad [0, \kappa], \\
\bar{z} = \int_{[\kappa, \infty)} b^1 F_2(z,0) h_2^o \, d\sigma_y d\sigma_x, & \gamma = x, y \quad \text{in} \quad (\kappa, \infty),
\end{cases}
\]

\[
X_b = \int_{[0, \kappa]} F_1(z) h_2^o \, d\sigma_y \quad \text{in} \quad [0, \kappa] \times \sigma_x(\kappa),
\]

\[
X_b = \int_{[\kappa, \infty)} F_2(z,0) h_2^o \, d\sigma_y \quad \text{in} \quad (\kappa, \infty) \times \sigma_x(\kappa)
\]

\[
X_b = 0 \quad \text{in} \quad I \times (I \setminus \sigma_x(\kappa)),
\]

\[
X_a = F_2(z,0, \alpha_1) \quad \text{in} \quad I \times (\tau_1, \tau_2],
\]

\[
X_a = F_2(z,0, \kappa) \quad \text{in} \quad I \times (\tau_2, \tau_3],
\]

\[
X_a = F_2(z, \alpha_2, \kappa) \quad \text{in} \quad [\alpha_2, \infty) \times (\tau_3, \tau_4],
\]

where \( \alpha_1 = \tau_x - \tau_1, \quad \alpha_2 = \tau_x - \tau_3. \)

\[
F_3(z, \rho, \xi) = \int_{\sigma_y} \left\{ \beta(\xi) \int_{\sigma_y} \chi F_2(z,0) \, d\sigma_x + \frac{\xi}{\rho} \int_{\sigma_y} \chi F_1(z) \, d\sigma_x \right\} d\sigma_y, \quad \beta(\xi) = \min(t, \xi).
\]

There is the delay argument \( t - \kappa \) in Eqs. (19) and (21). Therefore we can consider the problem (15)–(27), (4), (9) going in the consecutive order along the axis \( t \) by step \( \kappa. \)

Let \( t \in (0, \kappa]. \) From Eqs. (15), (19) and (21) it is easy to see, that functions \( \bar{x}, \gamma \) and \( X_b \) are expressed by the values of \( F_1(z) h_2^o \) and, consequently, are known. Thus \( y \) given by (16), is also known and the estimates (12), (13) are true. From (15), (9) and (22) we see, that terms, depending on \( F_2(z,0), \) in (24)–(26) for \( t \leq \kappa \) involve the function \( x(t - \tau_z, \tau_x - \tau_z). \) For \( t \leq \kappa \) in Eqs. (24) and (25) \( t - \tau_z \leq t, \quad \tau_x - \tau_z \leq \tau_3, \) while in Eq. (26) \( \tau_x - \tau_z \leq \tau_3. \) Therefore \( X_a \) involves \( x(t, \tau_x), \) and (17), where \( \tau_x \in (\tau_1, \tau_3], \) is the linear integral equation for \( x. \) When \( \tau_x \in (0, \tau_1], \) the right-hand side of the Eq. (18) is known, while for \( \tau_x > \tau_3 \) it involves the only unknown function \( x(t, \tau_3). \) When \( x(t, \tau_3) \) is obtained the right-hand side of (17) is also known. We will obtain \( x(t, \tau_3) \) by solving the Eq. (18) for \( \tau_x \in (\tau_1, \tau_3]. \)
Now we consider the Eq. (18) for $\tau_1, \tau_2 \in (\tau_1, \tau_3]$. Let $\Gamma_1 x$ be its right-hand side. By using the conditions of Theorem 1, we can prove, that the operator $\Gamma_1$ acts in the class $A$ of nonnegative continuous and bounded by the constant $a$ functions and is contractive. The norm of $x \in A$ is defined as $\sup |x|$. Therefore the equation $x = \Gamma_1 x$ has in $A$ a unique solution. We directly can prove the estimate (14).

Let $t \in (k, 2k]$. By using the known functions $x, y$, and $z$, when $t \leq k$, we obtain for $x$ an integral equation $x = \Gamma_2 x$ when $\tau_1 \in (\tau_1, \tau_3]$, and the certain expression when $\tau_1 \notin (\tau_1, \tau_3]$. The function $y$ is also known. By using the Banach's principle as above we can prove the unique solvability of equation $x = \Gamma_2 x$ and the estimates (12)–(14).

Going along the axis $t$ and using the same arguments as above we prove the unique solvability of the problem (15)–(26), (4), (9) and estimates (12)–(14) for $t \in I$. The functions $\bar{x}, \bar{y}, \bar{z}, D^y y, D^z z$ are continuous, while $D^k x$ is continuous except the lines $\tau_1, i = 1, 4$. This ends the proof of our theorem.

From (12)–(14) one can see, that for $q < 1$ the population vanishes as $t \to \infty$.

**NOTE.** We considered the case $\tau_3 - \tau_1 > k$. The opposite case can be considered in the same way.

2. The model taking into account females restoration intervals. We first briefly define the following notions used in [1]:

- $\tau_1$, $\tau_2$, and $\tau_3$ are ages of males, females and embryos, respectively;
- $\tau_1$ and $\tau_2$ are time passed after an abortion and a delivery, respectively, $t$ is time;
- $\sigma_x = (\tau_1, \tau_2)$, $0 < \tau_1 < \tau_2$ is the males' sexual activity interval;
- $\sigma_z = (0, \kappa_z)$, $0 < \kappa_z < \infty$ is the females' gestation interval, $\bar{\sigma}_z = [0, \kappa_z]$;
- $\sigma_u = (0, \kappa_u)$, $0 < \kappa_u < \infty$ and $\sigma_{\bar{u}} = (0, \kappa_{\bar{u}})$, $0 < \kappa_{\bar{u}} < \infty$ are the females' restoration intervals after abortions and deliveries, respectively, $\bar{\sigma}_u = [0, \kappa_{\bar{u}}]$, $\bar{\sigma}_{\bar{u}} = [0, \kappa_{\bar{u}}]$;
- $\sigma_{xm} = (\tau_1, \tau_2, \tau_3, \tau_4) = (\tau_1, \tau_2, \tau_3)$ are the males' fecundation and reproducibility intervals;
- $\sigma_{xm} = (0, \kappa_x)$ are the females' fecundation and reproducibility intervals;
- $\sigma_{xm} (\tau_1, \tau_2, \tau_3) = (\tau_1 + \tau_2, \tau_3, \tau_4, \tau_5, \tau_6)$, $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6 \in \bar{\sigma}_u$;
- $\sigma_{xm} (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = (\tau_1 + \kappa_x + \tau_0, \tau_2 + \kappa_x + \tau_0, \tau_3 + \kappa_x + \tau_0)$, $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6 \in \bar{\sigma}_u$;
- $\nu(t, \tau_y, \tau_y)$ is the females' fecundation rate;
- $\nu_1(t, \tau_y, \tau_y)$ and $\nu_2(t, \tau_y, \tau_y)$ are death rates of males, single and fecundated females, respectively;
- $\nu_1(t, \tau_y, \tau_y, \tau_y)$ and $\nu_2(t, \tau_y, \tau_y, \tau_y)$ are death rates of females from restoration intervals after abortions and deliveries, respectively;
- $\chi(t, \tau_y, \tau_y, \tau_y)$ is the abortions rate;
- $b_x(t, \tau_y, \tau_y)$ and $b_x(t, \tau_y, \tau_y)$ are the numbers of males and females, born from a female on a delivery.

$2^{1/2} D^y y, 2^{1/2} D^x x, 3^{1/2} D^z z, 3^{1/2} D^u u$ and $3^{1/2} D^u u$ represent directional derivatives along the positive direction of characteristics of operators $L_y = \partial / \partial t + \partial / \partial \tau_y$, $L_x = \partial / \partial t + \partial / \partial \tau_x$, $L_z = L_z + \partial / \partial \tau_z$, $L'_y = L_y + \partial / \partial \tau_u$ and $L'_u = L'_u + \partial / \partial \tau_u$, respectively.
\[ \sigma = \sigma_y \times \sigma_{xz}(\kappa_z), \quad I = (0, \infty), \quad I_b = [0, \infty), \quad E^y = \left\{ (t, \tau_y) \in I \times I \right\}, \quad E^x = \left\{ (t, \tau_x) \in I \times (I \setminus \cup_{i=1}^{6} \tau_i) \right\}, \]
\[ \tau_1 = \tau_{1x}, \quad \tau_2 = \tau_{1x} + \kappa_u, \quad \tau_3 = \tau_{1x} + \kappa_z + \kappa_v, \quad \tau_4 = \tau_{2x}, \quad \tau_5 = \tau_{2x} + \kappa_z + \kappa_u, \quad \tau_6 = \tau_{2x} + \kappa_z + \kappa_v, \]
\[ \kappa_u < \kappa_u < \kappa_z, \quad \kappa_u + \kappa_u + \kappa_z < \tau_{2x} - \tau_{1x}, \]
\[ E^z = \left\{ (t, \tau_y, \tau_x, \tau_z) \in I \times \sigma_y \times \sigma_{xz}(\tau_z) \times \sigma_z \right\}, \quad E^u = \left\{ (t, \tau_y, \tau_x, \tau_u) \in I \times \sigma_y \times \sigma_{xu}(\tau_u) \times \sigma_u \right\}, \]
\[ E^0 = \left\{ (t, \tau_y, \tau_x, \tau_u) \in I \times \sigma_y \times \sigma_{xu}(\tau_u) \times \sigma_u \right\}; \]
\[ \left[ x(t, \tau_i) \right] \text{ is a jump of the function } x \text{ at the line } \tau_x = \tau_i, \quad i = 1, 6; \]
\[ x^0(\tau_x), \quad y^0(\tau_y), \quad z^0(\tau_y, \tau_x, \tau_z), \quad u^0(\tau_y, \tau_x, \tau_u) \text{ and } v^0(\tau_y, \tau_x, \tau_u) \]
\[ \text{are the initial functions; } \]
\[ \Omega(\tau_x) = \begin{bmatrix} \sigma_y & \sigma_y \\ \sigma_y & \sigma_y \end{bmatrix}, \quad \Omega(\tau_x) = \begin{bmatrix} 0, \tau_x \in (\tau_1 + \kappa_z, \tau_1 + \kappa_z) \\ \tau_x \in (\tau_1, \tau_1 + \kappa_z) \end{bmatrix} \]
\[ \Omega(\tau_x) = \begin{bmatrix} 0, \tau_x \in (\tau_1 + \kappa_z, \tau_1 + \kappa_z) \\ \tau_x \in (\tau_1, \tau_1 + \kappa_z) \end{bmatrix} \]
\[ \text{The case } \kappa_u < \kappa_u < \kappa_z \text{ biologically is frequent and the inequality } \kappa_u + \kappa_u + \kappa_z < \tau_{4} - \tau_{1} \text{ means the multiple deliveries. The system (see [1])} \]
\[ D^{y}z = -yv^y \text{ in } E^y, \]
\[ D^{2}z = -zd^z, \quad d^z = v^z + \chi, \text{ in } E^z, \]
\[ D^{u}u = -uv^u \text{ in } E^u, \]
\[ D^{0}u = -uv^0 \text{ in } E^0, \]
\[ D^{x}x = -xd^x + X_a + X_b \text{ in } E^x, \]
\[ d^x = v^x + \begin{cases} n^{-1} \int ypdt_y, & n = \int ydt_y, \quad \tau_x \in \sigma_{xz}(0), \\ 0, \quad \tau_x \in \sigma_{xz}(0), \end{cases} \]
\[ X_a = \begin{bmatrix} \int u_{t_x = \kappa_u} dt_y, & \tau_x \in \sigma_{xu}(\kappa_u); \\ \sigma_y & \sigma_y \end{bmatrix}, \]
\[ X_b = \begin{bmatrix} \int v_{t_x = \kappa_u} dt_y, & \tau_x \in \sigma_{xu}(\kappa_u); \\ \sigma_y & \sigma_y \end{bmatrix}; \]
\[ \text{supplemented by the conditions} \]
\[ y_{t_o = 0} = y^0, \quad z_{t_o = 0} = z^0, \quad u_{t_o = 0} = u^0, \quad v_{t_o = 0} = v^0, \quad x_{t_o = 0} = x^0, \]
\[ y_{t_x = 0} = \int b^y z_{t_x = \kappa_z} dt_x, \quad x_{t_x = 0} = \int b^x z_{t_x = \kappa_z} dt_x, \]
\[ \left[ x_{t_x = \tau_i} = 0, \quad i = \overline{1, 6} \right] \]
governs the evolution of the population. The nonnegative demographic functions \( v^y, v^x, v^z, v^u, v^v, p, b^x, b^y, \chi \) and initial functions \( y^0, x^0, z^0, u^0, v^0 \) are assumed to be given. It is assumed also, that initial functions satisfy the reconcilable conditions, i.e. conditions (37)–(39) for \( t=0 \). As it follows from the biological meaning, unknown functions \( x, y, z, u \) must be nonnegative.

We note, that \((t, \tau_x)\) and \((t, \tau_y, \tau_x, \tau_z)\) are the arguments of functions \( d^x, X_a, X_b \) and \( d^z \), respectively.

Now we will consider the solvability problem of the model (28)–(39). Defining

\[
a = \sup_{0 \leq \tau_x \leq \tau_s} x^0, \quad p^* = \sup_{E^x} p, \quad \chi^* = \sup_{E^x} \chi, \quad d^x = \inf_{E^x} d^x, \quad \nu^x = \inf_{E^x} \nu^x, \quad \nu^y = \inf_{E^y} \nu^y.
\]

\[
b^* = \max_{i=x, y} \int_{\tau_x}^{\tau_s} b^i \, d\tau_x, \quad q = \frac{b^*}{a} \max_{i=x, y} \left( \int_{\tau_x}^{\tau_s} b^i \, d\tau_x \right).
\]

we can formulate the following statement.

**THEOREM 2.** Assume that:

1) \( \kappa_u < \kappa_u < \kappa_z \), \( \tau_4 - \tau_1 > \kappa_u + \kappa_u + \kappa_z \);

2) the nonnegative bounded functions \( b^x, b^y \) are continuous in \( t \) and piecewise continuous with respect to \( \tau = (\tau_y, \tau_x) \) in \( I \times \sigma \);

3) demographic functions \( p, \chi, v^x, v^y, v^z, v^u, v^v \) and initial functions \( x^0, y^0, z^0, u^0, v^0 \) satisfying the reconcilable conditions (37)–(39) for \( t=0 \), are continuous and bounded;

4) the constants \( a, p^*, \chi^*, d^x, \nu^x, \nu^y, b^*, q \), defined by (40), are finite, positive and such that:

\[
\int_{\tau_x}^{\tau_s} d\tau_x \leq \chi^* p^* \kappa_x \max(1, q / p^* b^*), \quad b^* p^* \exp\{-\kappa_x d^x\} \leq q \leq 1, \quad \sigma_x, \sigma_y, \sigma_z 
\]

\[
q / b^* + \chi^* p^* \kappa_x \max(1, q / p^* b^*) \leq \nu^x.
\]

Then the problem (28)–(30) has a unique nonnegative solution such that:

1) \( y, x, z, u, v \) are continuous functions in the domains \( I \times \bar{I}, I \times \bar{I}, I \times \sigma_y \times \sigma_x (\tau_x) \times \sigma_z \), \( I \times \sigma_y \times \sigma_x (\tau_u) \times \sigma_u \), and \( I \times \sigma_y \times \sigma_x (\tau_u) \times \sigma_u \), respectively;

2) \( \max(\sup x(t, 0), \sup y(t, 0)) \leq a q^{k+1}, k \tau_2 t < \tau > (k+1) \tau_2 x \),
\[
\begin{align*}
\{y^0(\tau_y - t) \exp\{-t/\nu^y\}, & \quad 0 \leq t < \tau_y < \infty, \\
\{a q^{k+1} \exp\{-t/\tau_y \nu^y\}, & \quad k \tau_2 t < \tau_y < (k+1) \tau_2 x, \tau_y \in I.
\end{align*}
\]
\[
x(t_x-t_0) \exp \left[ -t_a \xi \right], \quad 0 < t_0 \leq \tau_x, \quad \tau_x \in (0, \tau_2], \\
a, \quad 0 < t_0 \leq \tau_x, \quad \tau_x \in (\tau_2, \tau_6], \\
x(t_x-t_0) \exp \left[ -t_a \xi \right], \quad 0 < t_0 \leq \tau_x - \tau_6, \quad \tau_x > \tau_6, \\
\begin{cases}
a_0 \exp \left[ -\tau_6 \xi \right], & k \tau_2 < t - \tau_x \leq (k+1) \tau_2, \quad \tau_x \in (0, \tau_2] \\
a_0 \exp \left[ -(\tau_6 - \tau_2) \xi \right], & (k-1) \tau_2 < t - \tau_x \leq k \tau_2, \quad \tau_x > \tau_6, \\
\end{cases}
\]

\(k = 0, 1, 2, \ldots\)

Proof. The problem (28)-(39) can be solved by the method used in section 1. Assume \(\tau_0 = 0\), \(\tau_7 = \infty\).

By using the conditions of our theorem and the functions \(F_1(y), F_2(y, \mu)\), defined by (15), where \(X^y_{\alpha} = X^\mu_{\alpha} = 0\) for \(\gamma \neq x\), we can write the solution of the problem (28)-(39) in the integral form:

\[
y(t) = \begin{cases}
F_1(y), & y(t) = y^0(t_x - t), \quad 0 \leq t \leq t_y, \\
F_2(y, 0), & y(t - t_x), \quad 0 \leq t_y < t, \\
F_1(z), & z(t) = z^0(t_y, \tau_x - t, \tau_x - t), \quad 0 \leq t \leq \tau_z \leq \kappa_z, \\
F_2(z, 0), & z(t) = z(t_x - t, \tau_x - t), \quad 0 \leq \tau_z < t, \\
\end{cases}
\]

\[
x(t) = \begin{cases}
F_1(x), & x(t) = x^0(t_x - t), \quad 0 \leq t \leq \tau_x - \tau_i, \quad \tau_x \in (\tau_i, \tau_i + 1], \\
F_2(x, \tau_i), & t_x \in (\tau_i, \tau_i + 1], \quad t_x \in [0, \tau_6],
\end{cases}
\]

for \(i = 0, 6\),

\[
u(t) = \begin{cases}
F_1(u), & u(t) = u^0(t_y, \tau_x - t, \tau_x - t), \quad 0 \leq t \leq \tau_u \leq \kappa_u, \\
F_2(u, 0), & u(t) = u(t_x - t, \tau_x - t), \quad 0 \leq \tau_u < t,
\end{cases}
\]

where \(\tau_0 = (\eta, \eta + \tau_s - t), \quad \kappa_s = (\eta + t - \tau_s, \eta)\) for \(s = x, y\) and \(\tau_0 = (\eta, \tau_x, \eta + \tau_x - t, \eta + \tau_s - t), \quad \eta = (\eta, \tau_x, \eta + \tau_x - t, \eta + \tau_s - t), \quad \tau_0 = (\eta, \tau_s, \eta + \tau_x - t, \eta + \tau_s - t), \quad \eta = (\eta, \tau_s, \eta + \tau_x - t, \eta + \tau_s - t), \quad \tau_0 = (\eta, \tau_y, \eta + \tau_x - t, \eta + \tau_s - t), \quad \eta = (\eta, \tau_y, \eta + \tau_x - t, \eta + \tau_s - t), \quad \eta = (\eta, \tau_y, \eta + \tau_x - t, \eta + \tau_s - t), \quad \eta = \int \kappa_x dt_x.
\]

\[
u(t, \tau_y, \tau_x) = z(t_x = \kappa_x), \quad \nu(t, \tau_y, \tau_x) = \int \kappa_x dt_x.
\]

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Consider the functions $X_b, \bar{x}, \bar{y}, X_a$ in detail. Formulas (45), (48) show, that $X_b$ is known for $(t, \tau_x) \in (0, \kappa_v + \kappa_z) \times (\tau_3, \tau_5]$, while for $(t, \tau_x) \in (\kappa_v + \kappa_z, \infty) \times (\tau_3, \tau_5]$ it is represented by formula

$$X_b(t, \tau_x) = \int_{\sigma_y} \left( n^{-1} x_y \right) (\xi, \tau, \eta) f_1(t, \tau_x) \, d\tau,$$

(50)

where $\xi = t - \kappa_v - \kappa_z$, $\eta = \tau_x - \kappa_v - \kappa_z$, and $f_1$ is known. The functions $\bar{x}, \bar{y}$ are represented by (19), (20) where $\kappa$ must be replaced by $\kappa_z$. Eqs. (47) and (50) show that $X_a$ is known for $(t, \tau_x) \in (0, \kappa_u) \times (\tau_2, \tau_5]$ and

$$X_a(t, \tau_x) = \int_{\sigma_y} f_2(\xi_2, \tau_y, \eta_2) \left( \int_{\Omega_{\eta_2}} \left( n \chi_z \right)(\xi_2, \tau_y, \eta_2, \tau_z) \, d\tau_z \right) \, d\tau_y,$$

(51)

for $(t, \tau_x) \in (\kappa_u, \infty) \times (\tau_2, \tau_5]$, where $\xi_2 = t - \kappa_u$, $\eta_2 = \tau_x - \kappa_u$, and $f_2$ is known.

Using (45) from (20) with $\kappa$ replaced by $\kappa_z$ and (51) we can eliminate $z$ and, consequently, obtain for $y$ and $x$ the equations with delay arguments. We can solve these equations going along the axis $t$ with the step $\rho = \min(\kappa_u, \tau_1)$, because $x, y$ for $t \in [0, \rho]$ are known.

The continuity of the solution, mentioned in our theorem, and its differentiability along the respective characteristics are ensured by the conditions of theorem. The estimates (41)–(43) may be obtained directly by using the Gronwall's inequality. The estimate (43) shows, that for $q < 1$ the population vanishes as $t \to \infty$. This ends the proof.

REFERENCES