Positive Solutions of the Semipositone Neumann Boundary Value Problem

Johnny Henderson\textsuperscript{a} and Nickolai Kosmatov\textsuperscript{b}

\textsuperscript{a}Baylor University, Department of Mathematics
Waco, 76798-7328 TX, USA
\textsuperscript{b}University of Arkansas at Little Rock, Department of Mathematics and Statistics
Little Rock, 72204-1099 AR, USA
E-mail: Johnny.Henderson@baylor.edu
E-mail(corresp.): nxkosmatov@ualr.edu

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Abstract. In this paper we consider the Neumann boundary value problem at resonance

\[-u''(t) = f(t, u(t)), \quad 0 < t < 1, \quad u'(0) = u'(1) = 0.\]

We assume that the nonlinear term satisfies the inequality $f(t, z) + \alpha^2 z + \beta(t) \geq 0$, $t \in [0, 1]$, $z \geq 0$, where $\beta : [0, 1] \to \mathbb{R}_+$, and $\alpha \neq 0$. The problem is transformed into a non-resonant positone problem and positive solutions are obtained by means of a Guo–Krasnosel’skiǐ fixed point theorem.

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1 Introduction

We study the Neumann boundary value problem

\[-u''(t) = f(t, u(t)), \quad 0 < t < 1, \quad u'(0) = u'(1) = 0,\]  \hspace{1cm} (1.1)

with a sign-changing nonlinearity.

We will make the assumptions precise in the next section, we only mention now that the continuous function $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the inequality $f(t, z) \geq -\alpha^2 z - \beta(t)$ in $[0, 1] \times \mathbb{R}_+$, for some constant $\alpha \neq 0$ and a non-negative valued function $\beta(t)$.

One of the most frequently mentioned papers that stimulated the discussion of semipositone problems is the paper [7] by Miciano and Shivaji. The authors
of [7] used the bifurcation techniques to obtain multiple positive solutions for the Neumann problem. We only mention several among many results based on applications of a Guo–Krasnosel’skii fixed point theorem and fixed point index computations. In [10], Sun and Wei obtained positive solutions of the non-local boundary value problem

\[-u''(t) = f(t, u(t)), \quad 0 < t < 1,\]
\[u(0) = \alpha u(\eta), \quad u(1) = \beta u(\eta),\]

where the right side is a continuous function with \( f(t, u) + M \geq 0 \) for some \( M > 0 \). Lu [5] obtained multiple positive solutions for singular semipositone periodic boundary value problems. It should be mentioned that, in [5], the nonhomogeneous term depends on the first order derivative. In this regard, the results of [5] are similar to those obtained by Ma [6] who studied a fourth order semipositone boundary value problem

\[u^{(4)}(t) = \lambda f(t, u(t), u'(t)), \quad 0 < t < 1,\]
\[u(0) = u'(0) = u''(1) = u'''(1) = 0.\]

Other interesting results for second order boundary value problems can be found in [1, 4, 9, 13]. Semipositone boundary value problems of higher order have been studied in [2, 6, 11, 12] just to name a few. It seems, however, that resonant semipositone problems for ordinary differential equations have not been studied as extensively as their “invertible” counterparts. Nkashama and Santanilla [8] obtained nonpositive and nonnegative solutions of the Neumann problem using generalized Ambrosetti-Prodi conditions. Since we are unaware of results based on cone-theoretic methods, we believe that our study of the Neumann problem provides new results. We only treat the most basic case of (1.1) with a continuous right side.

2 Properties of Green’s Function

As a first step, we introduce \( g(t, z) = f(t, z) + \alpha^2 z \) to transform (1.1) into

\[-u''(t) + \alpha^2 u(t) = g(t, u(t)), \quad t \in (0, 1),\]

which we consider together with the boundary condition (1.2).

For \( \beta \in C[0, 1] \), the differential equation

\[-u''(t) + \alpha^2 u(t) = \beta(t), \quad 0 < t < 1,\]

satisfying the boundary condition (1.2) has a unique solution

\[u_0(t) = \int_0^t G(t, s) \beta(s) \, ds\]

with the Green function

\[G(t, s) = \frac{1}{\alpha \sinh \alpha} \begin{cases} \cosh \alpha (1 - t) \cosh \alpha s, & 0 \leq s \leq t \leq 1, \\ \cosh \alpha t \cosh \alpha (1 - s), & 0 \leq t \leq s \leq 1. \end{cases}\]
It is obvious that
\[ G(t, s) \leq G(s, s), \quad (t, s) \in [0, 1] \times [0, 1]. \]

If \( s \leq t \), then
\[
G(t, s) = \frac{1}{\alpha \sinh \alpha} \cosh \alpha (1 - t) \cosh \alpha s \\
\geq \frac{1}{\alpha \sinh \alpha} \cosh \alpha (1 - t) \cosh \alpha s \frac{\cosh \alpha (1 - s)}{\cosh \alpha} \\
\geq \frac{\cosh \alpha (1 - t)}{\cosh \alpha} G(s, s).
\]

Similarly, for \( t \leq s \),
\[
G(t, s) \geq \cosh \alpha t \cosh \alpha G(s, s).
\]

Combining the inequalities above, we obtain
\[
q(t) G(s, s) \leq G(t, s) \leq G(s, s), \quad (t, s) \in [0, 1] \times [0, 1],
\]
where
\[
q(t) = \frac{1}{\cosh \alpha} \min \{ \cosh \alpha t, \cosh \alpha (1 - t) \}.
\]

Also,
\[
L = \max_{t \in [0, 1]} \int_{0}^{1} G(t, s) \, ds = \frac{1}{\alpha^2}.
\]

For \( 0 < \gamma < 1/2 \),
\[
\int_{\gamma}^{1-\gamma} G(1-t, s) \, ds = \int_{\gamma}^{1-\gamma} G(1-t, 1-s) \, ds = \int_{\gamma}^{1-\gamma} G(t, s) \, ds.
\]

It suffices to consider
\[
\int_{\gamma}^{1-\gamma} G(t, s) \, ds
\]
\[
= \frac{1}{\alpha^2 \sinh \alpha} \left\{ (\sinh \alpha (1 - \gamma) - \sinh \alpha \gamma) \cosh \alpha t, \quad 0 \leq t \leq \gamma, \\
\sinh \alpha - \sinh \alpha \gamma (\cosh \alpha t + \cosh \alpha (1 - t)), \quad \gamma \leq t \leq 1/2, \right. \]

for \( t \in [0, 1/2] \), since the above function is symmetric about \( t = 1/2 \). Since it is increasing in \([0, 1/2]\),
\[
C = \max_{t \in [0, 1]} \int_{\gamma}^{1-\gamma} G(t, s) \, ds = \frac{1}{\alpha^2 \sinh \alpha} (\sinh \alpha - 2 \sinh \alpha \gamma \cosh \alpha/2).
\]

**Lemma 1.** Let \( \beta \in C[0, 1] \) and \( \beta(t) \geq 0 \) in \([0, 1]\), \( \beta(\tau) > 0 \) for some \( \tau \in [0, 1] \).

Then the inequality
\[
q(t) \geq \mu u_0(t), \quad t \in [0, 1],
\]
holds for
\[
\mu = \frac{\alpha \sinh \alpha}{\cosh^2 \alpha \int_{0}^{1} \beta(s) \, ds}.
\]
Proof. Note that

\[ u_0(t) = \int_0^1 G(t, s)\beta(s) \, ds \leq G(t, t) \int_0^1 \beta(s) \, ds. \]

Hence

\[ q(t) = \frac{1}{\cosh \alpha} \min \{ \cosh \alpha t, \cosh \alpha(1 - t) \} \]
\[ \geq \min \left\{ \frac{\cosh \alpha t}{\cosh \alpha}, \frac{\cosh \alpha(1 - t)}{\cosh \alpha} \right\} \frac{1}{\cosh \alpha} \max \{ \cosh \alpha t, \cosh \alpha(1 - t) \} \]
\[ = \frac{1}{\cosh^2 \alpha} \cosh \alpha t \cosh \alpha(1 - t) = \frac{\alpha \sinh \alpha}{\cosh^2 \alpha} G(t, t) \]
\[ = \mu G(t, t) \int_0^1 \beta(s) \, ds \geq \mu u_0(t) \]

for all \( t \in [0, 1] \). \( \square \)

Suppose that the function \( f \) in (1.1) satisfies

(A) \( f \in C([0, 1] \times \mathbb{R}_+, \mathbb{R}); \)

(B) there exists a function \( \beta \in C[0, 1], \beta(t) \geq 0 \) in \([0, 1], \beta(\tau) > 0 \) for some \( \tau \in [0, 1], \) and \( \alpha \in \mathbb{R}, \alpha \neq 0, \) such that

\[ f(t, z) + \alpha^2 z + \beta(t) \geq 0, \quad (t, z) \in [0, 1] \times \mathbb{R}_+. \]

We turn our attention to the equation

\[ -v''(t) + \alpha^2 v(t) = f_p(t, v(t) - u_0(t)), \quad t \in (0, 1), \quad (2.10) \]

where

\[ f_p(t, z) = \begin{cases} f(t, z) + \alpha^2 z + \beta(t), & (t, z) \in [0, 1] \times (0, \infty), \\ f(t, 0) + \beta(t), & (t, z) \in [0, 1] \times (-\infty, 0], \end{cases} \]

and impose the boundary conditions (1.2).

**Definition 1.** A positive solution of the boundary value problem (1.1), (1.2) is a function \( u \in C^2([0, 1] \times [0, 1], \mathbb{R}) \) satisfying (1.1), (1.2) and such that \( u(t) > 0 \) in \([0, 1] \).

The next lemma discusses the relationship between the problems (1.1), (1.2) and (2.10), (1.2) by means of a “shift” \( u \mapsto u + u_0 \) applied to the equation (2.1).

**Lemma 2.** The function \( u \) is a positive solution of the boundary value problem (1.1), (1.2) if and only if the function \( v = u + u_0 \), where \( u_0 \) is given by (2.2), is a solution of the boundary value problem (2.10), (1.2) satisfying \( v(t) > u_0(t) \) in \((0, 1)\).
In the Banach space $B = C[0,1]$ endowed with usual max-norm, we consider the operator
\[
Tv(t) = \int_0^1 G(t,s)f_p(s,v(s) - u_0(s)) \, ds,
\]
where $G(t,s)$ is given by (2.3). By (A), $T : B \to B$ is completely continuous.

Using the function $q$ defined by (2.5), we introduce the cone
\[
C = \{ v \in B : v(t) \geq q(t)\|v\|, \, t \in [0,1] \}.
\]

By (2.4), $T : C \to C$. One can easily confirm that a fixed point of $T$ in $C$ is a solution of (2.10), (1.2), and conversely. In particular, for $0 < \gamma < 1/2$,
\[
v(t) \geq \rho\|v\|, \quad t \in [\gamma, 1 - \gamma],
\]
where
\[
\rho = \min_{t \in [\gamma, 1-\gamma]} q(t) = \frac{\cosh \alpha \gamma}{\cosh \alpha}.
\]

The following is a fixed point theorem due to Guo and Krasnosel’skii.

**Theorem 1.** [3] Let $B$ be a Banach space and let $C \subset B$ be a cone in $B$. Assume that $\Omega_1, \Omega_2$ are open with $0 \in \Omega_1, \Omega_1 \subset \Omega_2$, and let
\[
T : C \cap (\Omega_2 \setminus \Omega_1) \to C
\]
be a completely continuous operator such that either
\begin{itemize}
  \item [(i)] $\|Tu\| \leq \|u\|$, $u \in C \cap \partial \Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in C \cap \partial \Omega_2$, or
  \item [(ii)] $\|Tu\| \geq \|u\|$, $u \in C \cap \partial \Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in C \cap \partial \Omega_2$.
\end{itemize}

Then $T$ has a fixed point in $C \cap (\Omega_2 \setminus \Omega_1)$.

### 3 Positive Solutions

To make use of Theorem 1, we introduce, following [11], the “height” functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by
\[
\phi(r) = \max \{ f_p(t, z - u_0(t)) : t \in [0,1], \, z \in [0,r] \}
\]
\[
\psi(r) = \min \{ f_p(t, z - u_0(t)) : t \in [\gamma, 1 - \gamma], \, z \in [\rho r, r] \}, \quad 0 < \gamma < 1/2.
\]

We present our main results.

**Theorem 2.** Assume that (A) and (B) hold. Suppose that there exist $r, R > 0$ such that $\frac{1}{\mu} < r < R$, where $\mu > 0$ satisfies (2.8), (2.9), and
\[
(C) \quad \phi(r) \leq \alpha^2 r \quad \text{and} \quad \psi(R) \geq \frac{\alpha^2 \sinh \alpha}{\sinh \alpha - 2 \sinh \alpha \cosh \alpha/2} R.
\]

Then the boundary value problem (1.1), (1.2) has at least one positive solution.
Proof. Let
\[ \Omega_1 = \{v \in \mathcal{B} : \|v\| < r\} \quad \text{and} \quad \Omega_2 = \{v \in \mathcal{B} : \|v\| < R\}. \]
For \( v \in \mathcal{C} \cap \partial \Omega_1 \), by Lemma 1, we have
\[ v(s) - u_0(s) \geq q(s)\|v\| - u_0(s) \geq (\mu r - 1)u_0(s) > 0, \quad s \in [0, 1]. \]
This implies that \( f_p(s, v(s) - u_0(s)) \leq \phi(r) \), for \( s \in [0, 1] \), \( 0 \leq v(s) \leq r \). Thus, by (2.6) and (C),
\[
\|Tv\| = \max_{t \in [0,1]} \int_0^1 G(t, s) f_p(s, v(s) - u_0(s)) \, ds \\
\leq \max_{t \in [0,1]} \int_0^1 G(t, s) \, ds \phi(r) = L\phi(r) \\
= \frac{1}{\alpha^2} \phi(r) \leq r.
\]
That is, \( \|Tv\| \leq \|v\| \) for all \( v \in \mathcal{C} \cap \partial \Omega_1 \).
Let \( v \in \mathcal{C} \cap \partial \Omega_2 \). Since \( R > r \), we have \( v(s) - u_0(s) \geq (\mu R - 1)u_0(s) \geq 0 \), \( s \in [0, 1] \). Then, for all \( s \in [\alpha, 1 - \alpha] \), we have, recalling (2.12),
\[
R \geq v(s) \geq q(s)\|v\| \geq \rho R.
\]
Hence \( f_p(s, v(s) - u_0(s)) \geq \psi(R) \), for \( s \in [\gamma, 1 - \gamma] \), \( \gamma R \leq v(s) \leq R \). Then, by (2.7) and (C),
\[
\|Tv\| = \max_{t \in [0,1]} \int_0^1 G(t, s) f_p(s, v(s) - u_0(s)) \, ds \\
\geq \max_{t \in [0,1]} \int_{\gamma}^{1-\gamma} G(t, s) f_p(s, v(s) - u_0(s)) \, ds \\
\geq \max_{t \in [0,1]} \int_{\gamma}^{1-\gamma} G(t, s) \, ds \psi(R) = C\psi(R) \\
= \frac{1}{\alpha^2 \sinh \alpha} (\sinh \alpha - 2 \sinh \alpha \gamma \cosh \alpha/2) \psi(R) \geq R.
\]
That is, \( \|Tv\| \geq \|v\| \) for all \( v \in \mathcal{C} \cap \partial \Omega_2 \).
By Theorem 1, there exists a fixed point \( v_0 \in \mathcal{C} \) of (2.11), which, equivalently, is a positive solution of the positone problem (2.10), (1.2). Moreover, \( u(t) = v_0(t) - u_0(t) \geq (\mu r - 1)u_0(t) > 0 \) in \([0, 1]\). By Lemma 2, \( u \) is a positive solution of the sign-changing problem (1.1), (1.2). \( \square \)

The next result can be shown along similar lines.

**Theorem 3.** Assume that (A) and (B) hold. Suppose that there exist \( r, R > 0 \) such that \( \frac{1}{\mu} < r < R \), where \( \mu > 0 \) satisfies (2.8), (2.9), and

\[
(D) \quad \phi(R) \leq \alpha^2 R \quad \text{and} \quad \psi(r) \geq \frac{\alpha^2 \sinh \alpha}{\sinh \alpha - 2 \sinh \alpha \gamma \cosh \alpha/2} r.
\]

Then the boundary value problem (1.1), (1.2) has at least one positive solution.
References


