On the Modulus of the Selberg Zeta-Functions in the Critical Strip

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Abstract. We investigate the behavior of the real part of the logarithmic derivatives of the Selberg zeta-functions $Z_{PSL(2,Z)}(s)$ and $Z_C(s)$ in the critical strip $0 < \sigma < 1$. The functions $Z_{PSL(2,Z)}(s)$ and $Z_C(s)$ are defined on the modular group and on the compact Riemann surface, respectively.

Keywords: Selberg zeta-function, modular group, compact Riemann surface, Riemann zeta-function, critical strip.

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1 Introduction

Let $s = \sigma + it$ denote a complex variable. We start with the definition and some properties of the Riemann zeta-function. For $\sigma > 1$, the Riemann zeta-function is given by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and can be analytically continued to the whole complex plane, except for a simple pole at $s = 1$ with residue 1. Trivial zeros of $\zeta(s)$ are located at the negative even integers. The remaining, the so-called non-trivial zeros, lie on the critical strip $0 < \sigma < 1$. The Riemann zeta-function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin \frac{\pi s}{2} \zeta(1-s),$$

or $\xi(s) = \xi(1-s)$, where $\xi(s) = \frac{1}{2} \pi^{-s/2} \Gamma(s/2) \zeta(s)$, and $\Gamma(s)$ denotes the Euler gamma-function. The function $\xi(s)$ is an entire function whose zeros are the non-trivial zeros of $\zeta(s)$, see [19, §II].
In the paper [11], it was proved the following relation between functions \( \zeta(s) \) and \( \xi(s) \).

**Theorem 1.** The functions \( \zeta(s) \) and \( \xi(s) \) satisfy, for \( |t| \geq 8 \) and \( \sigma < 1/2 \), the inequality
\[
\text{Re} \frac{\zeta'(s)}{\zeta(s)} < \text{Re} \frac{\xi'(s)}{\xi(s)}.
\]

Sondow and Dumitrescu proved in [17] the following theorem for the function \( \xi(s) \).

**Theorem 2.** The function \( \xi(s) \) is increasing in modulus along every horizontal half-line lying in any open right half-plane that contains no its zeros. Similarly, the modulus decreases on each horizontal half-line in any zero-free, open left half-plane.

In the same paper, the following reformulation for the Riemann hypothesis that all non-trivial zeros of \( \zeta(s) \) lie on the line \( \sigma = 1/2 \) was given.

**Theorem 3.** The following statements are equivalent:

I. If \( t \) is any fixed real number, then \(|\xi(\sigma + it)|\) is increasing for \( 1/2 < \sigma < \infty \).

II. If \( t \) is any fixed real number, then \(|\xi(\sigma + it)|\) is decreasing for \(-\infty < \sigma < 1/2 \).

III. The Riemann hypothesis is true.

Later, Theorem 3 was reproved in [11] in a slightly different way.

Related properties of the functions \( \zeta(s) \) and \( \xi(s) \) in the critical strip were also investigated in [15].

In this paper, we ask whether Selberg zeta-functions have similar properties as the Riemann-zeta function has in Theorems 1 - 3. Note that, for Selberg zeta-functions, the analogue of the Riemann hypothesis is usually valid. We consider Selberg zeta-functions for cocompact and modular subgroups.

Let \( \mathbb{H} \) be the upper half-plane, and \( \Gamma \) be a subgroup of \( \text{PSL}(2, \mathbb{R}) \). Let \( \Gamma \backslash \mathbb{H} \) be a hyperbolic Riemann surface of finite area. The Selberg zeta-function \( Z(s) \) is defined [5], for \( \sigma > 1 \), by
\[
Z(s) = \prod_{\{P\}} \prod_{k=0}^{\infty} (1 - N(P)^{-s-k}),
\]
where \( \{P\} \) runs through all primitive hyperbolic conjugacy classes of \( \Gamma \), and \( N(P) = \alpha^2 \) if the eigenvalues of \( P \) are \( \alpha \) and \( \alpha^{-1} \), \( |\alpha| > 1 \). The Selberg zeta-function has a meromorphic continuation to the whole complex plane [5].

If \( \Gamma \backslash \mathbb{H} \) is a compact Riemann surface of genus \( g \geq 2 \), we use the notation \( Z(s) = Z_C(s) \). If \( \Gamma = \text{PSL}(2, \mathbb{Z}) \), then we denote \( Z(s) = Z_{\text{PSL}(2, \mathbb{Z})}(s) \). Similarly, as the Riemann zeta-function, the Selberg zeta-function \( Z_{\text{PSL}(2, \mathbb{Z})}(s) \) has a meromorphic continuation to the whole complex plane, and satisfies the symmetric functional equation [8]
\[
\Xi(s) = \Xi(1-s),
\]

where
\[ \Xi(s) = Z_{PSL(2,\mathbb{Z})}(s)Z_{id}(s)Z_{ell}(s)Z_{par}(s), \]
and
\begin{align*}
Z_{id}(s) &= \left(\frac{(2\pi)^s}{\Gamma(s)}\right)^{1/6} (\Gamma_2(s))^{1/3}, \\
Z_{par}(s) &= \frac{\pi^s}{\Gamma(s)\zeta(2s)\Gamma(s+1/2)2^s}, \\
Z_{ell}(s) &= \Gamma\left(\frac{s}{2}\right)^{-1/2} \Gamma\left(\frac{s+1}{2}\right)^{1/2} \Gamma\left(\frac{s}{3}\right)^{-2/3} \Gamma\left(\frac{s+2}{3}\right)^{2/3}.
\end{align*}

The function $\Gamma_2(s)$ is called the double Barnes gamma-function, and is defined by the canonic product
\[ \frac{1}{\Gamma_2(s+1)} = (2\pi)^{s/2} \exp\left\{ \frac{-s}{2} - \frac{(\gamma_0 + 1)s^2}{2} \right\} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{s}{k}\right)^k \exp\left(-s + \frac{s^2}{2k}\right) \right\}, \]
where $\gamma_0$ denotes the Euler constant. The function $\Gamma_2(s)$ satisfies the relations
\[ \Gamma_2(1) = 1, \quad \Gamma_2(s+1) = \frac{\Gamma_2(s)}{\Gamma(s)}, \quad \Gamma_2(n+1) = \frac{1^2\cdot2^2\cdots n^2}{(n!)^n}, \]
see, for example [1], [16] or [20].

The function $\Xi(s)$ is an entire function of order 2, and has zeros at the points $s = 1/2 + ir_n$, $n \geq 0$, where $r_n = \sqrt{\lambda_n - \frac{1}{4}}$, and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ are the eigenvalues of the Laplace operator [3], [7]. The function $Z_{PSL(2,\mathbb{Z})}(s)$ has poles and zeros at the following points [6]:

**Poles of $Z_{PSL(2,\mathbb{Z})}(s)$:**
1. $s = 0$; order 1,
2. $s = 1/2 - k$, $k \geq 0$; order 1.

**Zeros of $Z_{PSL(2,\mathbb{Z})}(s)$:**
1. $s = 1$; order 1,
2. $s = -6k - j$, $k \geq 0$, $j = 1, 2, 3, 4, 6$; order $2k + 1$,
3. $s = -6k - 5$, $k \geq 0$; order $2k + 3$,
4. $s = \rho/2$, where $\rho$ are non-trivial zeros of $\zeta(s)$,
5. $s = 1/2 \pm ir_n$, $n \geq 0$.

We prove the following theorem.

**Theorem 4.** There exists a positive number $C$ such that, for $t > C$ and $0 < \sigma < 1/2$,
\[ \text{Re} \frac{\Xi'(s)}{\Xi(s)} < 0. \]
Furthermore, if we assume the Riemann hypothesis for $\zeta(s)$, then there exists a positive number $C_1$ such that
\[ \text{Re} \frac{Z_{PSL(2,\mathbb{Z})}'(s)}{Z_{PSL(2,\mathbb{Z})}(s)} < \text{Re} \frac{\Xi'(s)}{\Xi(s)}. \]
for $t > C_1$ and $0 < \sigma < 1/4$. Conversely, if
\[ \frac{Z'_{\text{PSL}(2,\mathbb{Z})}(s)}{Z_{\text{PSL}(2,\mathbb{Z})}(s)} < 0 \]
for $t > C_1$ and $0 < \sigma < 1/4$, then the function $\zeta(s)$, for $t > C_1$, has zeros only for $\sigma = 1/2$.

Theorem 4 is proved in the next section. Below, we formulate a couple of corollaries of Theorem 4. We also want to mention that a part of assertions of Theorem 4 can be obtained following the proof of Theorem 6.1 in [12].

**Corollary 1.** For a fixed sufficiently large $t$, the function $|\Xi(\sigma + it)|$ is decreasing for $0 < \sigma < 1/2$, and is increasing for $1/2 < \sigma < 1$ with respect to $\sigma$.

**Corollary 2.** If the Riemann hypothesis is true for $\zeta(s)$, then, for a sufficiently large fixed $t$, the function $|Z_{\text{PSL}(2,\mathbb{Z})}(\sigma + it)|$ is decreasing for $0 < \sigma < 1/4$ with respect to $\sigma$.

Proofs of Corollaries 1 and 2 follow from Lemma 1, functional equation $\Xi(s) = \Xi(1-s)$ and equality $\Xi(\overline{s}) = \overline{\Xi(s)}$.

We return to Selberg zeta-functions attached to compact Riemann surfaces. The function $Z_C(s)$ is an entire function of order 2 [4, §2.4, Theorem 2.4.25] and satisfies the functional equation [4, §2.4, Theorem 2.4.12]
\[ Z_C(s) = f(s)Z_C(1-s), \]
where
\[ f(s) = \exp\left(4\pi(g-1)\int_0^{s-1/2} v \tan(\pi v) dv\right), \]
and $g \geq 2$ is the genus of a Riemann surface. The above functional equation is equivalent to $M(s) = M(1-s)$, where
\[ M(s) = Z_C(s) \exp\left(2\pi(g-1)\int_0^{1/2-s} v \tan \pi v dv\right). \]

The Selberg zeta-function $Z_C(s)$ has trivial zeros at $s = 1, 0, -1, -2, \ldots$, non-trivial zeros on the critical line $\sigma = 1/2$ and also, possibly, on the interval $(0, 1)$ of the real axis, see [4, §2.4, Theorem 2.4.11] and [13]. In this sense, the analogue of the Riemann hypothesis holds for $Z_C(s)$. Moreover, the following statement is true.

**Theorem 5.** There exists a positive number $B$ such that the functions $Z_C(s)$ and $M(s)$, for $t > B$, $0 < \sigma < 1/2$, satisfy the inequality
\[ \Re\frac{Z'_C(s)}{Z_C(s)} < \Re\frac{M'(s)}{M(s)} < 0. \]

Note that a part of Theorem 5 is proved in [9], namely,

$$\text{Re} \frac{Z_C'(s)}{Z_C(s)} < 0$$

for $-c \leq \sigma \leq 1/2$ and $t \geq t_0 > 0$, where $c > 0$ is an arbitrary constant, and $t_0$ is a constant depending on $c$.

A couple of corollaries follow from Theorem 5 for functions $Z_C(s)$ and $M(s)$.

**Corollary 3.** For a fixed and sufficiently large $t$, the function $|M(\sigma + it)|$ is decreasing for $0 < \sigma < 1/2$, and is increasing for $1/2 < \sigma < 1$.

**Corollary 4.** For a fixed and sufficiently large $t$, the function $|Z_C(\sigma + it)|$ is decreasing for $0 < \sigma < 1/2$.

Proofs of Corollaries 3 and 4 are the same as proofs of Corollaries 1 and 2, and Theorem 5 is proved in Section 3.

## 2 Proof of the Theorem 4

Before the proof of Theorem 4, we state one lemma.

**Lemma 1.** (a) Let $f$ be a holomorphic function in an open domain $D$ and not identically zero. Let us also suppose $\text{Re} \frac{f'(s)}{f(s)} < 0$ for all $s \in D$ such that $f(s) \neq 0$. Then $|f(s)|$ is strictly decreasing with respect to $\sigma$ in $D$, i.e., for each $s_0 \in D$, there exists $\delta > 0$ such that $|f(s)|$ is strictly monotonically decreasing with respect to $\sigma$ on the horizontal interval from $s_0 - \delta$ to $s_0 + \delta$.

(b) Conversely, if $|f(s)|$ is decreasing with respect to $\sigma$ in $D$, then $\text{Re} \frac{f'(s)}{f(s)} \leq 0$ for all $s \in D$ such that $f(s) \neq 0$.

The proof of the lemma is given in [11].

**Remark 1.** Of course, the analogous results hold for monotonically increasing $|f(s)|$ and $\text{Re} \frac{f'(s)}{f(s)} > 0$.

Now we prove Theorem 4.

**Proof of Theorem 4.** First we prove that

$$\text{Re} \frac{Z_{PSL(2,\mathbb{Z})}'(s)}{Z_{PSL(2,\mathbb{Z})}(s)} < \text{Re} \frac{\Xi'(s)}{\Xi(s)}, \quad t > C_1 > 0, \ 0 < \sigma < 1/4.$$

From the equality $\Xi(s) = Z_{PSL(2,\mathbb{Z})}'(s)Z_{id}(s)Z_{ell}(s)Z_{par}(s)$, we find that

$$\frac{\Xi'(s)}{\Xi(s)} = \frac{Z_{PSL(2,\mathbb{Z})}'(s)}{Z_{PSL(2,\mathbb{Z})}(s)} + Z_{id}'(s) + Z_{ell}'(s) + Z_{par}'(s) = \frac{Z_{PSL(2,\mathbb{Z})}'(s)}{Z_{PSL(2,\mathbb{Z})}(s)} + U(s).$$
Hence, to complete the proof it is sufficient to show that

$$\text{Re}U(s) > 0, \quad t > C_1 > 0, \quad 0 < \sigma < 1/4.$$  

By (1.1), we obtain

$$U(s) = a_0 + \frac{1}{4} \left( \Psi \left( \frac{s}{2} + \frac{1}{2} \right) - \Psi \left( \frac{s}{2} \right) \right) + \frac{2}{9} \left( \Psi \left( \frac{s}{3} + \frac{2}{3} \right) - \Psi \left( \frac{s}{3} \right) \right)$$

$$+ \frac{1}{3} \Psi_2(s) - \frac{7}{6} \Psi(s) - \Psi \left( s + \frac{1}{2} \right) - 2 \frac{\zeta'}{\zeta}(2s),$$

where $a_0 = \frac{1}{6} \log 2\pi + \log \frac{\pi}{2} = 0.757 \ldots$, $\Psi(s) = \Gamma'(s)/\Gamma(s)$ and $\Psi_2(s) = \Gamma'_2(s)/\Gamma_2(s)$.

To prove the inequality $\text{Re}U(s) > 0$, we need to investigate the behavior of the functions $\Psi(s)$, $\Psi_2(s)$ and $\zeta'(2s)/\zeta(2s)$ in the region $0 < \sigma < 1/4$ and $t > C_1 > 0$. For the function $\Psi(s)$, the estimate [10]

$$\Psi(s) = \log s - \frac{1}{2s} + O \left( \frac{1}{|s|^2} \right), \quad |s| \to \infty, \quad |\arg s| \leq \pi - \delta < \pi,$$

holds. From this, we deduce that

$$\text{Re}\Psi(s) = \log t + O \left( \frac{1}{t} \right), \quad t \to \infty, \quad |\arg s| < \pi. \quad (2.2)$$

It is known [21] that, for $-s \notin \mathbb{N}$

$$\frac{\Gamma_2'(s + 1)}{\Gamma_2(s + 1)} = \frac{\Psi_2(s + 1)}{\Psi_2(s + 1)} = \frac{1 - \log 2\pi}{2} + (\gamma_0 + 1)s - \sum_{k=1}^{\infty} \left( \frac{k}{k + s} - 1 + \frac{s}{k} \right)$$

$$= -\frac{1 + \log 2\pi}{2} + s - s\Psi(s).$$

This and (2.2) show that

$$\text{Re}\Psi_2(s) = -\frac{3 + \log 2\pi}{2} + \sigma + (1 - \sigma)\text{Re}\Psi(s - 1) + t\text{Im}\Psi(s - 1)$$

$$= -\frac{3 + \log 2\pi}{2} + \sigma + (1 - \sigma)\log t + t \left( \pi - \arctan \left( \frac{t}{\sigma - 1} \right) \right) + O \left( \frac{1}{t} \right)$$

$$= -\frac{3 + \log 2\pi}{2} + \sigma + (1 - \sigma)\log t + t \arctan \left( \frac{t}{\sigma} \right) + O \left( \frac{1}{t} \right) \quad (2.3)$$

for $0 < \sigma < 1/4$ and $t > C_1 > 0$.

From the formula [2]

$$\xi(s) = \xi(0) \prod_{\rho} \left( 1 - \frac{s}{\rho} \right),$$

we obtain that

$$\frac{\xi'}{\xi}(s) = \sum_{\rho} \frac{1}{s - \rho}.$$
where the summation runs over all non-trivial zeros of the Riemann zeta-function taken in conjugate pairs and in order of increasing imaginary parts. If $\rho = \beta + i\gamma$, then

$$\text{Re} \frac{\zeta'(s)}{\zeta(s)} = \sum_{\beta + i\gamma} \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2}.$$  

If we assume the Riemann hypothesis, i.e., $\beta = 1/2$, then $\text{Re}(\xi(s)'/\xi(s)) > 0$ for $\sigma > 1/2$, and $\text{Re}(\xi(s)'/\xi(s)) < 0$ for $\sigma < 1/2$.

On the other hand, from the equation

$$\xi(s) = (s - 1)\pi^{-s/2} \Gamma(s/2 + 1) \zeta(s)$$

we get

$$\frac{\xi'(s)}{\xi(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2} \Psi \left( \frac{s}{2} + 1 \right) - \frac{1}{2} \log \pi + \frac{1}{s - 1}.$$  

This yields that, for $\sigma > 1/2$,

$$\text{Re} \frac{\xi'(s)}{\xi(s)} > \frac{1}{2} \log t - \frac{1}{2} \log 2\pi + O \left( \frac{1}{t} \right),$$  

and, for $\sigma < 1/2$,

$$-\text{Re} \frac{\xi'(s)}{\xi(s)} > \frac{1}{2} \log t - \frac{1}{2} \log 2\pi + O \left( \frac{1}{t} \right).$$  

In view of (2.1), (2.2), (2.3) and (2.5), we find that for $t \to \infty$,

$$\text{Re} U(s) = a_0 - \frac{13}{6} \log t + \frac{1}{3} \text{Re} \Psi_2(s) - 2\text{Re} \frac{\zeta'(s)}{\zeta(s)}(2s) + O \left( \frac{1}{t} \right)$$

$$= \log \frac{\pi}{2} + \frac{\sigma}{3} - \frac{1}{2} - \frac{2\sigma + 11}{6} \log t + \frac{t}{3} \arctan \left( \frac{t}{\sigma} \right) - 2\text{Re} \frac{\zeta'(s)}{\zeta(s)}(2s) + O \left( \frac{1}{t} \right)$$

$$> \frac{t}{3} \arctan \left( \frac{t}{\sigma} \right) - \frac{5 + 2\sigma}{6} \log t + c(\sigma) + O \left( \frac{1}{t} \right),$$

where $a_0 = \frac{1}{6} \log 2\pi + \log \frac{\pi}{2}$ and $c(\sigma) = \log \frac{1}{2} + \frac{\sigma}{3} - \frac{1}{2}$. This shows that there exists a constant $C_1 > 0$ such that $\text{Re} U(s)$ is positive for $t > C_1$ and $0 < \sigma < 1/4$. Hence, for $t > C$ and $0 < \sigma < 1/4$,

$$\text{Re} \frac{Z'_{\text{PSL}(2,\mathbb{Z})}(s)}{Z_{\text{PSL}(2,\mathbb{Z})}(s)} < \text{Re} \frac{\xi'(s)}{\xi(s)}.$$  

We note that the restriction of $\sigma < 1/4$ is due to the zeros of the function $\zeta(2s)$.

Now we prove that

$$\text{Re} \frac{\Xi'(s)}{\Xi(s)} < 0.$$
for \( t > C_1 \) and \( 0 < \sigma < 1/2 \). The function \( \Xi(s) \) is an entire function of order two. It has a canonical product expansion [14], [18]

\[
\Xi(s) = e^{as^2 + bs + cs^2} \prod_{\hat{\rho}} \left( 1 - \frac{s}{\hat{\rho}} \right) e^{s/\hat{\rho} + (1/2)(s/\hat{\rho})^2},
\]

(2.7)

where \( \hat{\rho} \) runs over the nonzero roots of \( \Xi(s) \), and \( a, b, c, \) and \( n \) are constants. This implies

\[
\Xi'(s) = 2as + b + \frac{n}{s} + \sum_{\hat{\rho}} \left( \frac{s^2}{\hat{\rho}^2} - \frac{1}{\hat{\rho}} + \frac{1}{s - \hat{\rho}} \right).
\]

If \( \hat{\rho} = 1/2 + ir_n, n \geq 0 \), then the latter sum splits into two parts: for those \( \hat{\rho} \) for which the numbers \( 1/2 + ir_n \) are real, and for those \( \hat{\rho} \) for which the numbers \( 1/2 + ir_n \) are complex. There are only a finite number of real numbers \( 1/2 + ir_n \). Then

\[
\text{Re} \left( \frac{\Xi'(s)}{\Xi(s)} \right) = 2a\sigma + b + \frac{n\sigma}{\sigma^2 + t^2} + \sum_{n > n_0} \frac{\sigma(1/4 - r_n^2) + tr_n}{(1/4 - r_n^2)^2 + r_n^2}
\]

\[
+ \sum_{n > n_0} \frac{1/2}{1/4 + r_n^2} + \sum_{n > n_0} \frac{\sigma - 1/2}{(\sigma - 1/2)^2 + (t - r_n)^2}
\]

\[
+ \sum_{0 \leq n \leq n_0} \left( \frac{\sigma}{(1/2 + ir_n)^2} + \frac{1}{(1/2 + ir_n)^2} + \frac{\sigma - 1/2 - ir_n}{\sigma - 1/2 - ir_n + t^2} \right). \tag{2.8}
\]

We see that the sum

\[
\sum_{n > n_0} \frac{\sigma(1/4 - r_n^2) + tr_n}{(1/4 - r_n^2)^2 + r_n^2} = \frac{\sigma(1/4 - r_{n_0+1}^2) + tr_{n_0+1}}{(1/4 - r_{n_0+1}^2)^2 + r_{n_0+1}^2} + \sum_{n > n_0+1} \frac{\sigma(1/4 - r_n^2) + tr_n}{(1/4 - r_n^2)^2 + r_n^2}
\]

is positive and unbounded as \( t \to \infty \). Then, from equation (2.8), it follows that there exists a number \( C > 0 \) such that

\[
\text{Re} \left( \frac{\Xi'(s)}{\Xi(s)} \right) > 0
\]

for \( t > C \) and \( 1/2 < \sigma < 1 \). By a note after Lemma 1, for fixed \( t > C \), the function \( |\Xi(\sigma + it)| \) is monotonically increasing as a function of \( \sigma, 0 < \sigma < 1/2 \). In view of the functional equation \( \Xi(s) = \Xi(1 - s) \) and \( \Xi(s) = \Xi(s) \), the function \( |\Xi(\sigma + it)| \) is monotonically decreasing for \( t > C \) as a function of \( \sigma, 1/2 < \sigma < 1 \). So, the real part of its logarithmic derivative is negative, and the second assertion of the theorem holds.

The statement that if

\[
\text{Re} \left( \frac{Z'_{PSL(2, Z)}(s)}{Z_{PSL(2, Z)}(s)} \right) < 0
\]

for \( t > C_1 \) and \( 0 < \sigma < 1/4 \), then the Riemann hypothesis is true, follows straightforward from Lemma 1 and the fact that the function \( Z_{\text{PSL}(2, \mathbb{Z})}(s) \) has zeros \( s = \rho/2 \), where \( \rho \) are non-trivial zeros of \( \zeta(s) \).

Recall that

\[
U(s) = a_0 + \frac{1}{4} \left( \psi \left( \frac{s}{2} + \frac{1}{2} \right) - \psi \left( \frac{s}{2} \right) \right) + \frac{2}{9} \left( \psi \left( \frac{s}{3} + \frac{2}{3} \right) - \psi \left( \frac{s}{3} \right) \right) \\
+ \frac{1}{3} \psi_2(s) - \frac{7}{6} \psi(s) - \psi \left( s + \frac{1}{2} \right) - 2 \frac{\zeta'(2s)}{\zeta(2s)},
\]

where \( a_0 = \frac{1}{6} \log 2\pi + \log \pi = 0.757 \ldots \).

**Corollary 5.** If \( 0 < \sigma < 1/4 \), then

\[-\text{Re} \frac{Z_{\text{PSL}(2, \mathbb{Z})}(s)}{Z_{\text{PSL}(2, \mathbb{Z})}(s)} > \text{Re}(U(s)) > \frac{t}{3} \arctan \left( \frac{t}{\sigma} \right) - \frac{5 + 2\sigma}{6} \log t + c(\sigma) + O \left( \frac{1}{t} \right) \]

holds. If \( 1/2 < \sigma < 1 \), then

\[-\text{Re} \frac{Z_{\text{PSL}(2, \mathbb{Z})}(s)}{Z_{\text{PSL}(2, \mathbb{Z})}(s)} < \text{Re}(U(s)) < \frac{t}{3} \arctan \left( \frac{t}{\sigma} \right) - \frac{5 + 2\sigma}{6} \log t + c(\sigma) + O \left( \frac{1}{t} \right) \]

holds, where \( c(\sigma) = \log \frac{1}{2} + \frac{\sigma}{3} - \frac{1}{2} \) and \( t \to \infty \).

**Proof.** The first part of the corollary follows from the fact \( \text{Re} (\Xi'/\Xi(s)) < 0 \), \( 0 < \sigma < 1/2 \), and inequality (2.6). The second part is obtained analogically. \( \square \)

### 3 Proof of Theorem 5

**Proof of Theorem 5.** Recall that the Selberg zeta-function attached to compact Riemann surfaces satisfies the functional equation \( M(s) = M(1 - s) \), where

\[
M(s) = Z_C(s) \exp \left( 2\pi (g - 1) \int_0^{1/2-s} v \tan \pi v \, dv \right).
\]

The function \( M(s) \) is an entire function of order two, and it has the same form of canonical product expansion (2.7) as the function \( \Xi(s) \). So, for \( t > t_0 > 0 \), the function \( |M(s)| \) is monotonically decreasing with respect to \( 0 < \sigma < 1/2 \).

Let

\[
l(s) = \exp \left( \int_0^{1/2-s} v \tan \pi v \, dv \right).
\]
To complete the proof, we need to show that

$$\text{Re} \left( \frac{L'(s)}{L(s)} \right) > 0$$

for $0 < \sigma < 1/2$, and $t > \hat{t}_0$. By elementary calculation, we obtain

$$\text{Re} \left( \frac{L'(s)}{L(s)} \right) = \text{Re} \left\{ \left( s - \frac{1}{2} \right) \tan \pi \left( \frac{1}{2} - s \right) \right\}$$

$$= \frac{t(1 - e^{-4\pi t})}{e^{-4\pi t} - 2e^{-2\pi t} \cos 2\pi \sigma + 1} + \left( \sigma - \frac{1}{2} \right) \frac{2e^{-2\pi t} \sin 2\pi \sigma}{e^{-4\pi t} - 2e^{-2\pi t} \cos 2\pi \sigma + 1}$$

$$= t(1 + o(1)),$$

as $t \to \infty$. Taking $B = \max(t_0, \hat{t}_0)$ completes the proof.

In the same way the following corollary follows.

**Corollary 6.** If $0 < \sigma < 1/2$, then

$$-\text{Re} \frac{Z_C'(s)}{Z_C(s)} > 2\pi (g - 1) \cdot t \cdot (1 + O(e^{-2\pi t})), \quad t \to \infty,$$

holds. If $1/2 < \sigma < 1$, then

$$-\text{Re} \frac{Z_C'(s)}{Z_C(s)} < 2\pi (g - 1) \cdot t \cdot (1 + O(e^{-2\pi t})), \quad t \to \infty,$$

holds.

**Proof.** Proof is the same as that for Corollary 5. \qed

### 4 Some remarks on the Riemann zeta-function

In this section, we present some remarks on the Riemann zeta-function $\zeta(s)$, which could have been obtained proving Theorems 4 and 5.

Let, as above, $\rho = \beta + i\gamma$ be non-trivial zeros of $\zeta(s)$. Recall that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_\rho \frac{1}{s - \rho},$$

where the summation is over all non-trivial zeros of the Riemann zeta-function taken in conjugate pairs in order of increasing imaginary parts. Also,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2} \psi \left( \frac{s}{2} + 1 \right) - \frac{1}{2} \log \pi + \frac{1}{s - 1}.$$

Comparing the latter equalities with

$$\frac{\zeta'(s)}{\zeta(s)} = b - \frac{1}{s - 1} - \frac{1}{2} \psi \left( \frac{s}{2} + 1 \right) + \sum_\rho \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right),$$

where \( b = \log 2\pi - 1 - \gamma_0/2 \), we have [19] that

\[
\sum_{\rho} \frac{1}{\rho} = 1 + \frac{\gamma_0}{2} - \frac{1}{2} \log 4\pi.
\]

The inequalities (2.4) and (2.5) give the bounds for the real part of the logarithmic derivative of the Riemann zeta-function in the half-planes \( \sigma < 1/2 \) and \( \sigma > 1/2 \), respectively. Assuming the Riemann hypothesis, allows to construct more precise bounds. For this we need some lemmas.

**Lemma 2.** Let \( N(T) \) be the number of zeros of \( \zeta(s) \) in the rectangle \( 0 < \sigma < 1, \ 0 < t < T \). Then, as \( T \to \infty \),

\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + R(T),
\]

where \( R(T) = O(\log T) \). If the Riemann hypothesis is true, then \( R(T) = O\left(\frac{\log T}{\log \log T}\right) \).

The proof of the lemma can be found, for example, in [19].

**Lemma 3.** For \( t > 1 \), the inequality

\[
\arctan t < \frac{\pi}{2} - \frac{1}{2t}
\]

holds.

**Proof.** We have that

\[
\frac{\pi}{2} = \int_0^\infty \frac{dx}{1 + x^2} = \int_0^t \frac{dx}{1 + x^2} + \int_t^\infty \frac{dx}{1 + x^2} > \arctan t + \int_t^\infty \frac{dx}{x^2 + x^2} = \arctan t + \frac{1}{2t}.
\]

\( \square \)

**Lemma 4.** Let \( \rho_1 = 1/2 + i\gamma_1, \gamma_1 = 14.134725 \ldots \), be the first non-trivial zero of \( \zeta(s) \). Then

\[
\sum_{\gamma > 0} \frac{1}{1/4 + (\gamma - t)^2} > \frac{1}{2} \log \frac{t}{\gamma_1} + O\left(\frac{1}{t}\right)
\]

and

\[
\sum_{\gamma > 0} \frac{1}{1/4 + (\gamma + t)^2} > \frac{1}{8\pi t} \log \frac{t}{2\pi} + O\left(\frac{1}{t}\right)
\]

as \( t \to \infty \).
Proof. By Lemma 2, summing by parts, we get
\[
\sum_{\gamma > 0} \frac{1}{1/4 + (\gamma - t)^2} = \int_{\gamma_1}^\infty \frac{1}{1/4 + (u - t)^2} d\left(\frac{u}{2\pi} \log \frac{u}{2\pi} - \frac{u}{2\pi} + R(u)\right) + O\left(\frac{1}{t^2}\right)
\]
\[
= \frac{1}{2\pi} \int_{\gamma_1}^\infty \log(u/2\pi) du \cdot \frac{1}{1/4 + (u - t)^2} + O\left(\int_{\gamma_1}^\infty \log u \cdot \frac{1}{1/4 + (u - t)^2}\right) + O\left(\frac{1}{t^2}\right)
\]
\[
\frac{1}{2\pi} \int_{\gamma_1}^t \log(u/2\pi) du \cdot \frac{1}{1/4 + (u - t)^2} + \frac{1}{2\pi} \int_t^\infty \log(u/2\pi) du \cdot \frac{1}{1/4 + (u - t)^2} + O\left(\frac{1}{t}\right)
\]
\[
= \frac{1}{2\pi} \log \frac{\gamma_1}{2\pi} \int_{\gamma_1}^t \frac{du}{1/4 + (u - t)^2} + \frac{1}{2\pi} \log \frac{t}{2\pi} \int_t^\infty \frac{du}{1/4 + (u - t)^2} + O\left(\frac{1}{t}\right)
\]
\[
> \frac{1}{2\pi} \frac{\log \frac{\gamma_1}{2\pi}}{\gamma_1} \log \frac{\gamma_1}{2\pi} + \frac{1}{2\pi} \log \frac{t}{2\pi} + O\left(\frac{1}{t}\right)
\]
\[
> \frac{1}{2} \log \frac{t}{\gamma_1} + O\left(\frac{1}{t}\right), \quad t \to \infty,
\]
where the last inequality was obtained using that $-\pi/2 \leq \arctan v \leq \pi/2$. This proves the first part of the lemma.

Similar arguments and Lemma 3 show that
\[
\sum_{\gamma > 0} \frac{1}{1/4 + (\gamma + t)^2} \leq \frac{1}{2\pi} \int_{\gamma_1}^t \frac{\log(u/2\pi) du}{1/4 + (u + t)^2} + \frac{1}{2\pi} \int_t^\infty \frac{\log(u/2\pi) du}{1/4 + (u + t)^2} + O\left(\frac{1}{t}\right)
\]
\[
> \frac{1}{2\pi} \log \frac{\gamma_1}{2\pi} \left(2\arctan 4t - 2\arctan(2(t - \gamma_1))\right) + \frac{1}{2\pi} (\pi - 2\arctan 4t) \log \frac{t}{2\pi}
\]
\[
+ O\left(\frac{1}{t}\right) > \frac{1}{8\pi t} \log \frac{t}{2\pi} + O\left(\frac{1}{t}\right), \quad t \to \infty.
\]

It is well known that
\[
\text{Re} \frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} - \frac{\sigma - 1}{(\sigma - 1)^2 + t^2}
\]
\[
- \frac{1}{2} \text{Re} \left(\psi\left(\frac{s}{2} + 1\right)\right) + \frac{1}{2} \log \pi.
\]

Assume the Riemann hypothesis. Then, in view of Lemma 4, we obtain
\[
\sum_{\gamma} \frac{1}{(\sigma - 1/2)^2 + (t - \gamma)^2} > \sum_{\gamma} \frac{1}{1/4 + (t - \gamma)^2}
\]
\[
= \sum_{\gamma > 0} \frac{1}{1/4 + (t - \gamma)^2} + \sum_{\gamma > 0} \frac{1}{1/4 + (t + \gamma)^2}
\]
\[
> \frac{1}{2} \log \frac{t}{\gamma_1} + \frac{1}{8\pi t} \log \frac{t}{2\pi} + O\left(\frac{1}{t}\right), \quad t \to \infty.
\]

Using this and (2.2), we find that

\[-\text{Re} \frac{\zeta'}{\zeta}(s) > -\frac{1}{2} \left( \sigma - \frac{3}{2} \right) \log t - \frac{\sigma - 1/2}{8\pi t^2} \log \frac{t}{2\pi} - \frac{1}{2} \log 2\pi + \frac{\sigma - 1/2}{2} \log \gamma_1 + O \left( \frac{1}{t} \right),\]

for $0 < \sigma < 1/2$, and

\[-\text{Re} \frac{\zeta'}{\zeta}(s) < -\frac{1}{2} \left( \sigma - \frac{3}{2} \right) \log t - \frac{\sigma - 1/2}{8\pi t^2} \log \frac{t}{2\pi} - \frac{1}{2} \log 2\pi + \frac{\sigma - 1/2}{2} \log \gamma_1 + O \left( \frac{1}{t} \right),\]

for $1/2 < \sigma < 1$ as $t \to \infty$.

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References


