Weighted Discrete Universality of the Riemann Zeta-Function

Antanas Laurinčikas\textsuperscript{a}, Darius Šiaučiūnas\textsuperscript{b} and Gediminas Vadeikis\textsuperscript{a}

\textsuperscript{a}Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University
Naugarduko str. 24, LT-03225 Vilnius, Lithuania

\textsuperscript{b}Institute of Regional Development, Šiauliai University
P. Višinskio str. 25, LT-76351 Šiauliai, Lithuania

E-mail (corresp.): darius.siauciunas@su.lt
E-mail: antanas.laurincikas@mif.vu.lt
E-mail: gediminas.vadeikis@mif.vu.lt

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Abstract. It is well known that the Riemann zeta-function is universal in the Voronin sense, i.e., its shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. The universality of $\zeta(s)$ is called discrete if $\tau$ take values from a certain discrete set. In the paper, we obtain a weighted discrete universality theorem for $\zeta(s)$ when $\tau$ takes values from the arithmetic progression $\{kh : k \in \mathbb{N}\}$ with arbitrary fixed $h > 0$. For this, two types of $h$ are considered.

Keywords: approximation of analytic functions, Mergelyan theorem, Riemann zeta-function, universality, weak convergence.

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1 Introduction

The Riemann zeta-function $\zeta(s)$, $s = \sigma + it$,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

since Riemann’s and even Euler’s times surprises mathematicians by the extensive field of applications and denseness of the set of its values. It is well
known the role of $\zeta(s)$ in the theory of distribution of prime numbers and in other problems of arithmetic, however, we, in this paper, prefer the denseness properties of $\zeta(s)$.

In the second decade of the last century, H. Bohr discovered [4] that the function $\zeta(s)$ takes every non-zero value infinitely many times in the strip $\{s \in \mathbb{C} : 1 < \sigma < 1 + \delta\}$ with any $\delta > 0$. H. Bohr and R. Courant proved [5] that, for fixed $\sigma$, $\frac{1}{2} < \sigma \leq 1$, the set

$$\{\zeta(\sigma + it) : t \in \mathbb{R}\}$$  \hspace{1cm} (1.1)

is dense in $\mathbb{C}$. S.M. Voronin significantly generalized the above results. He obtained [25] that the set

$$\{(\zeta(s_1 + i\tau), \ldots, \zeta(s_n + i\tau)) : \tau \in \mathbb{R}\}$$

with any fixed numbers $s_1, \ldots, s_n$, $\frac{1}{2} < \text{Re} s_k < 1$, $1 \leq k \leq n$, and $s_k \neq s_m$ for $k \neq m$, and the set

$$\left\{ \left( \zeta(s + i\tau), \zeta'(s + i\tau), \ldots, \zeta^{(n-1)}(s + i\tau) \right) : \tau \in \mathbb{R} \right\}$$

with every fixed $s$, $\frac{1}{2} < \sigma < 1$, are dense in $\mathbb{C}^n$. However, a much more important merit of Voronin is his so-called universality theorem for the function $\zeta(s)$ [26]. This theorem asserts that a wide class of analytic functions can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. For a modern version of the Voronin universality theorem, it is convenient to use the following notation. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D$ with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on $K$ that are analytic in the interior of $K$.

Then the following theorem is true.

**Theorem 1.** Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$  

Here $\text{meas} A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. By Theorem 1, the set of shifts $\zeta(s + i\tau)$ approximating a given function from $H_0(K)$ has a positive lower density, thus, it is infinite. Also, Theorem 1 can be considered as an infinite-dimensional version of the Bohr-Courant theorem on denseness of the set (1.1). The proof of Theorem 1 is given in [1] (in slightly different form), and in [9], [11], [24].

Theorem 1 is of continuous type: $\tau$ in $\zeta(s + i\tau)$ can take arbitrary real values. Also, a discrete version of Theorem 1 is known when $\tau$ takes values from a certain discrete set. Let $h > 0$ be a fixed number.

**Theorem 2.** Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\} > 0.$$
Here \#A denotes the cardinality of the set A. The proof of Theorem 2 can be found in [22] and [1]. For shifts \( \zeta(s + ik\alpha h) \) with fixed \( \alpha, 0 < \alpha < 1 \), Theorem 2 is given in [6]. In [15, 21] and [7, 8, 13, 16], more general shifts of Dirichlet L-functions and Riemann zeta-function, respectively, were considered. We note that discrete universality theorems for zeta-functions sometimes are more convenient for practical applications, an example of this is the paper [3].

In [10], a weighted version of Theorem 1 was proposed. Let \( w(t) \) be a function of bounded variation on \([T_0, \infty)\) with some \( T_0 > 0 \) such that the variation \( V_N^b w \) on \([a, b]\) satisfies the inequality \( V_N^b w \leq cw(a) \) with a certain constant \( c > 0 \) for any subinterval \([a, b] \subset [T_0, \infty)\). Let

\[
U_T = U(T, w) = \int_{T_0}^{T} w(t)dt,
\]

and let \( \lim_{T \to \infty} U(T, w) = +\infty \). Moreover, let \( I_A \) denote the indicator function of the set \( A \). Then we have the following generalization of Theorem 1.

**Theorem 3.** Suppose that the function \( w(t) \) satisfies the above hypotheses. Let \( K \in \mathcal{K} \) and \( f(s) \in H_0(K) \). Then, for every \( \varepsilon > 0 \),

\[
\lim \inf_{T \to \infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I_{\left\{ \tau : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\}}(\tau)d\tau > 0.
\]

To be precise, in [10], Theorem 3 was proved under a certain additional hypothesis on the function \( w(t) \) which is a weighted version of the classical Birkhoff-Khintchine ergodic theorem. In [18], this technical hypothesis was removed. A generalization of Theorem 3 for Matsumoto zeta-functions was given in [12]. In [17], a weighted discrete universality theorem with the sequence \( \{k^\alpha h\}, \quad 0 < \alpha < 1 \), for the periodic zeta-function was obtained.

The aim of this paper is a weighted discrete universality theorem for the Riemann zeta-function. Let \( w(t) \) be a real non-negative function having a continuous derivative on \([\frac{1}{2}, \infty)\) such that

\[
\lim_{N \to \infty} V_N = +\infty, \quad V_N = \sum_{k=1}^{N} w(k), \quad \int_{1}^{N} u|w'(u)|du \ll V_N, \quad N \to \infty.
\]

Denote by \( W \) the class of functions \( w(t) \) satisfying the above hypotheses. Suppose that \( h \) is a fixed positive number.

**Theorem 4.** Suppose that \( w(t) \in W \). Let \( K \in \mathcal{K} \) and \( f(s) \in H_0(K) \). Then, for every \( \varepsilon > 0 \),

\[
\lim \inf_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\left\{ k : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\}}(k) > 0.
\]

For example, the function \( w(t) = \frac{\sin(\log t) + 1}{t} \) is not monotonically decreasing and \( w(t) \in W \).

Theorem 4 has the following modification.

Theorem 5. Suppose that \( w(t) \in W \). Let \( K \in \mathcal{K} \) and \( f(s) \in H_0(K) \). Then the limit
\[
\lim_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I \{ k : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon \} (k) > 0
\]
exists for all but at most countably many \( \varepsilon > 0 \).

For proving of the above universality theorems, we will apply the probabilistic approach.

2 Limit theorems

We remind that \( D = \{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \} \), and by \( H(D) \) denote the space of analytic functions on \( D \) endowed with the topology of uniform convergence on compacta. The space \( H(D) \) is metrisable. There exists a sequence of compact subsets \( \{ K_l : l \in \mathbb{N} \} \subset D \) such that \( D = \bigcup_{l=1}^{\infty} K_l \), \( K_l \subset K_{l+1} \) for all \( l \in \mathbb{N} \), and if \( K \subset D \) is a compact set, then \( K \subset K_l \) for some \( l \in \mathbb{N} \). For \( g_1, g_2 \in H(D) \), define
\[
\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K} |g_1(s) - g_2(s)|}.
\]
Then \( \rho \) is a metric on \( H(D) \) which induces its topology of uniform convergence on compacta.

Denote by \( B(X) \) the Borel \( \sigma \)-field of the space \( X \), and, for \( A \in B(H(D)) \), define
\[
P_N(A) = P_{N,w,h}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I \{ k : (p-ikh : p \in \mathbb{P}) \in A \} (k).
\]
In this section, we will consider the weak convergence of \( P_{N,w,h} \) as \( N \to \infty \).

We say that \( h > 0 \) is of type 1 if \( \exp \{ \frac{2\pi m}{h} \} \) is an irrational number for all \( m \in \mathbb{Z} \setminus \{0\} \), and \( h > 0 \) is of type 2 if \( h \) is not of type 1. We will examine separately the cases of types 1 and 2.

As usual, we start with one topological structure. Let \( \gamma = \{ s \in \mathbb{C} : |s| = 1 \} \) and \( \Omega = \prod_p \gamma_p \), where \( \gamma_p = \gamma \) for all primes \( p \). By the Tikhonov theorem, the torus \( \Omega \) with the product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on \( (\Omega, B(\Omega)) \), the probability Haar measure \( m_H \) can be defined, and this gives the probability space \( (\Omega, B(\Omega), m_H) \). Let \( \mathbb{P} \) be the set of all prime numbers, and let \( \omega(p) \) denote the projection of \( \omega \in \Omega \) to the circle \( \gamma_p \), \( p \in \mathbb{P} \). For \( A \in B(\Omega) \), define
\[
Q_N(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I \{ k : (p^{-ikh} : p \in \mathbb{P}) \in A \} (k).
\]

Lemma 1. Suppose that \( w(t) \in W \) and \( h \) is of type 1. Then \( Q_N \) converges weakly to the Haar measure \( m_H \) as \( N \to \infty \).
Proof. We apply the Fourier transform method. Let $g_N(k), k = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, be the Fourier transform of $Q_N$. Then we have that
\[
g_N(k) = \int_{\Omega} \prod_p^* \omega^{k_p}(p)dQ_N,
\]
where the sign "*" means that only a finite number of integers $k_p$ are distinct from zero. Thus, by the definition of $Q_N$,
\[
g_N(k) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) \prod_p^* p^{-ik_p h} = \frac{1}{V_N} \sum_{k=1}^{N} w(k) \exp \left\{ -ikh \sum_p^* k_p \log p \right\},
\]
(2.1)
Obviously,
\[
g_N(0) = 1. \tag{2.2}
\]
If $k \neq 0$, then
\[
\sum_p^* k_p \log p \neq 0,
\]
since the logarithms of prime numbers are linearly independent over the field of rational numbers. Thus,
\[
\exp \left\{ -ih \sum_p^* k_p \log p \right\} \neq 1. \tag{2.3}
\]
Indeed, if inequality (2.3) is not true, then
\[
\sum_p^* k_p \log p = \frac{2\pi r}{h}, \quad \prod_p^* p^{-k_p} = \exp \left\{ \frac{2\pi r}{h} \right\}
\]
with some $r \in \mathbb{Z} \setminus \{0\}$. However, the left-hand side of this equality is a rational number, and we arrive to the contradiction that $h$ is of type 1. Thus, (2.3) is true, and we find that, for $u \geq 1$,
\[
\sum_{k \leq u} \exp \left\{ -ikh \sum_p^* k_p \log p \right\} = \frac{\exp \left\{ -ih \sum_p^* k_p \log p \right\} - \exp \left\{ i(\lfloor u \rfloor + 1)h \sum_p^* k_p \log p \right\}}{1 - \exp \left\{ -ih \sum_p^* k_p \log p \right\}} \overset{\text{def}}{=} \Sigma(u).
\]
Hence, in view of (2.1), for $k \neq 0$,
\[
g_N(k) = \frac{w(N) \Sigma(N)}{V_N} - \frac{1}{V_N} \int_1^N \Sigma(u)w'(u)du.
\]
Since the function $\Sigma(u)$ is bounded by a constant not depending of $u$, we find that, for $k \neq 0$,
\[
\lim_{N \to \infty} g_N(k) = 0.
\]
Lemma 1 implies a weighted discrete universality theorem for absolutely convergent Dirichlet series. Let \( \theta > \frac{1}{2} \) be a fixed number, and

\[
v_n(m) = \exp \left\{ -\left( \frac{m}{n} \right)^{\theta} \right\}, \quad m, n \in \mathbb{N},
\]

\[
\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}, \quad \zeta_n(s, \omega) = \sum_{m=1}^{\infty} \frac{\omega(m)v_n(m)}{m^s},
\]

where

\[
\omega(m) = \prod_{p^\alpha \mid m \atop p^{\alpha + 1} \nmid m} \omega_\alpha(p), \quad m \in \mathbb{N}.
\]

Then the series for \( \zeta_n(s) \) and \( \zeta_n(s, \omega) \) are absolutely convergent for \( \sigma > \frac{1}{2} \) \[11\]. From this, it follows that the function \( u_n : \Omega \rightarrow H(D), u_n(\omega) = \zeta_n(s, \omega) \), is continuous. Let \( R_n = m_Hu_n^{-1} \), where

\[
R_n(A) = m_Hu_n^{-1}(A) = m_H(u_n^{-1}A), \quad A \in \mathcal{B}(H(D)).
\]

Moreover, let

\[
P_{N,n}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k)I_{\{k: \zeta_n(s+ikh) \in A\}}(k), \quad A \in \mathcal{B}(H(D)).
\]

It is not difficult to see that \( P_{N,n} = Q_Nu_n^{-1} \). This, the continuity of \( u_n \) and Lemma 1 lead to

**Lemma 2.** Suppose that \( w(t) \in W \) and \( h \) is of type 1. Then \( P_{N,n} \) converges weakly to \( R_n \) as \( N \rightarrow \infty \).

The weak convergence of \( P_{N,n} \) is a starting point for proving the weak convergence for \( P_N \) as \( N \rightarrow \infty \). The investigation of \( P_N \) also requires an approximation of \( \zeta(s) \) by \( \zeta_n(s) \). Let \( l_n(s) = \frac{s}{\theta} \Gamma \left( \frac{s}{\theta} \right) n^s \), where \( \Gamma(s) \) is the Euler gamma-function. Then \[11\], for \( \sigma > \frac{1}{2} \), the integral representation

\[
\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z)l_n(z)\frac{dz}{z} \quad (2.4)
\]

is true. Using the well-known estimates

\[
\int_{1/2}^{T} |\zeta(\sigma + it)|^2 dt \ll T, \quad \int_{1/2}^{T} |\zeta'(\sigma + it)|^2 dt \ll T,
\]

we find that, for \( \frac{1}{2} < \sigma < 1 \) and \( \tau \in \mathbb{R} \),

\[
\int_{1/2}^{T} |\zeta(\sigma + it + i\tau)|^2 dt \ll T(1 + |\tau|)
\]
and
\[ \int_{1/2}^{T} |\zeta'(\sigma + it + i\tau)|^2 \, dt \ll T(1 + |\tau|). \]

These estimates together with Gallagher lemma, see, for example, [20, Lemma 1.4], give, for \( \frac{1}{2} < \sigma < 1 \) and \( \tau \in \mathbb{R} \), the bound
\[ \sum_{k=1}^{N} |\zeta(\sigma +ikh + i\tau)|^2 \ll \int_{1/2}^{(N+1/2)h} |\zeta(\sigma + it)|^2 \, dt \]
\[ + \left( \int_{1/2}^{(N+1/2)h} |\zeta'(\sigma +it + i\tau)|^2 \, dt \int_{1/2}^{(N+1/2)h} |\zeta'(\sigma +it + i\tau)|^2 \, dt \right)^{1/2} \ll N(1 + |\tau|). \]

Hence, for the same \( \sigma \) and \( \tau \),
\[ \sum_{k=1}^{N} w(k) |\zeta(\sigma +ikh + i\tau)|^2 \ll w(N) \sum_{k=1}^{N} |\zeta(\sigma +ikh + i\tau)|^2 + (1 + |\tau|) \]
\[ \times \int_{1}^{N} u|w'(u)| \, du \ll Nw(N)(1 + |\tau|) + V_N(1 + |\tau|) \ll V_N(1 + |\tau|), \quad (2.5) \]

because
\[ Nw(N) = \sum_{k=1}^{N} w(k) + \int_{1}^{N} \left( \sum_{k \leq u} 1 \right) w'(u) \, du \ll V_N. \]

Let \( K \subset D \) be a compact set. Then (2.4), (2.5), the residue theorem and Cauchy integral formula imply the equality
\[ \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) \sup_{s \in K} |\zeta(s +ikh) - \zeta_n(s +ikh)| = 0. \quad (2.6) \]

Now, (2.6) together with the definition of the metric \( \rho \) yields the following lemma.

**Lemma 3.** Suppose that \( w(t) \in W \). Then the equality
\[ \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) \rho(\zeta(s +ikh), \zeta_n(s +ikh)) = 0 \]
is true for every fixed \( h > 0 \).

Now, we are in position to prove a weighted discrete limit theorem for the function \( \zeta(s) \). On the probability space \( (\Omega, \mathcal{B}(\Omega), m_H) \), define the \( H(D) \)-valued random element \( \zeta(s, \omega) \) by the Euler product
\[ \zeta(s, \omega) = \prod_p \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1}. \]

The latter product, for almost all \( \omega \in \Omega \), is uniformly convergent on compact subsets of the strip \( D \) [11]. Denote by \( P_\zeta \) the distribution of the random element \( \zeta(s, \omega) \), i.e.,

\[
P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega) \in A), \quad A \in \mathcal{B}(H(D)).
\]

**Theorem 6.** Suppose that \( w(t) \in W \) and \( h > 0 \) is of the type 1. Then \( P_N \) converges weakly to \( P_\zeta \) as \( N \to \infty \). Moreover, the support of \( P_\zeta \) is the set 

\[
S = \{ g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.
\]

**Proof.** We will prove that \( R_n \), as \( n \to \infty \), converges weakly to a certain probability measure \( P \), and that \( P_N \), as \( N \to \infty \), also converges weakly to \( P \).

Let \( \theta_N \) be a random variable defined on a certain probability space with probability measure \( \mu \) and having the distribution

\[
\mu(\theta_N = kh) = \frac{w(k)}{V_N}, \quad k = 1, \ldots, N.
\]

Moreover, let \( Y_{N,n} = Y_{N,n}(s) \) be an \( H(D) \)-valued random element defined by

\[
Y_{N,n}(s) = \zeta_n(s + i\theta_N),
\]

and let \( Y_n = Y_n(s) \) be an \( H(D) \)-valued random element with the distribution \( R_n \). Then, by Lemma 2,

\[
Y_{N,n} \overset{\mathcal{D}}{\to} Y_n. \tag{2.7}
\]

Using the absolute convergence of the series for \( \zeta_n(s) \), it can be proved by a method of [11] that the family of probability measures \( \{ R_n : n \in \mathbb{N} \} \) is tight, i.e., for every \( \varepsilon > 0 \), there exists a compact set \( K = K(\varepsilon) \subset H(D) \) such that

\[
R_n(K) > 1 - \varepsilon
\]

for all \( n \in \mathbb{N} \). Hence, by the Prokhorov theorem [2], this family is relatively compact. Therefore, each sequence of \( \{ R_n \} \) contains a subsequence \( \{ R_{n_r} \} \) weakly convergent, as \( r \to \infty \), to a certain probability measure \( P \) on \((H(D), \mathcal{B}(H(D)))\).

In other words,

\[
Y_{n_r} \overset{\mathcal{D}}{\to} P. \tag{2.8}
\]

Define one more \( H(D) \)-valued random element

\[
X_N = X_N(s) = \zeta(s + i\theta_N).
\]

Then the application of Lemma 3 gives, for \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \limsup_{N \to \infty} \mu(\rho(X_N(s), Y_{N,n}(s)) \geq \varepsilon)
\]

\[
= \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I\{k : \rho(\zeta(s + ikh), \zeta_n(s + ikh)) \geq \varepsilon\}(k)
\]

\[
\leq \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{\varepsilon V_N} \sum_{k=1}^{N} w(k) \rho(\zeta(s + ikh), \zeta_n(s + ikh)) = 0.
\]
This equality, (2.7) and (2.8) show that all hypotheses of Theorem 4.2 of [2] are satisfied, therefore,
\[ X_N \xrightarrow{P_{N \to \infty}} P, \quad (2.9) \]
or \( P_N \) converges weakly to \( P \) as \( N \to \infty \). Moreover, in virtue of (2.9), the measure \( P \) is independent of the sequence \( Y_{n_r} \). Since the family \( \{ R_n \} \) is relatively compact, from this, we obtain that \( R_n \) converges weakly to \( P \) as \( n \to \infty \). Thus, \( P_N \), as \( N \to \infty \), converges weakly to the limit measure \( P \) of \( R_n \) as \( n \to \infty \).

However, by the proof of a limit theorem for
\[ \text{meas} \{ \tau \in [0, T] : \zeta(s + i\tau) \in A \}, \quad A \in B(H(D)), \]
it is known [11] that \( R_n \), as \( n \to \infty \), converges weakly to \( P_\zeta \), and the support of \( P_\zeta \) is the set \( S \). Therefore, the same statement is also true for \( P_N \), and the theorem is proved. \( \Box \)

The case of \( h \) of type 2 is a more complicated. We must construct a new probability space different from \( (\Omega, B(\Omega), m_H) \). We will index by \( h \) the notation related to \( h \) of type 2.

Now suppose that \( h > 0 \) is of type 2. Then there exists the smallest \( m_0 \in \mathbb{N} \) such the number \( \exp \left\{ \frac{2\pi m_0}{h} \right\} \) is rational. We put \( \exp \left\{ \frac{2\pi m_0}{h} \right\} = \frac{a}{b}, \quad a, b \in \mathbb{N}, \quad (a, b) = 1 \).

Define the set
\[ \mathbb{P}_0 = \left\{ p \in \mathbb{P} : \frac{a}{b} = \prod_{p \in \mathbb{P}} p^{\alpha_p} \text{ with } \alpha_p \neq 0 \right\}. \]

Denote by \( \Omega_h \) the closed subgroup of \( \Omega \) generated by the element \( \{ p^{-ih} : p \in \mathbb{P} \} \). By Lemma 1 of [14], if \( h \) is of type 2, then
\[ \Omega_h = \{ \omega \in \Omega : \omega(a) = \omega(b) \}. \]

On \( (\Omega_h, B(\Omega_h)) \), the probability Haar measure \( m_H \) exists, and we obtain the probability space \( (\Omega_h, B(\Omega_h), m_H) \). By (3.1) of [14], we have that the characters \( \chi \) of the group \( \Omega_h \) are of the form
\[ \chi(\omega) = \prod_{p \in \mathbb{P} \setminus \mathbb{P}_0}^* \omega^{k_p}(p) \prod_{p \in \mathbb{P}_0} \omega^{k_p + l\alpha_p}(p), \quad l \in \mathbb{Z}. \quad (2.10) \]

Now, we are ready to prove an analogue of Lemma 1 for \( h \) of type 2. For \( A \in B(\Omega_h) \), define
\[ Q_{N,h}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\{ k: (p^{-ih} : p \in \mathbb{P}) \in A \}}(k). \]

**Lemma 4.** Suppose that \( h \) is of type 2. Then \( Q_{N,h} \) converges weakly the Haar measure \( m_H \) as \( N \to \infty \).
Proof. In view of (2.10), we have that the Fourier transform $g_{N,h}(k)$, $k = (k_p : k_p \in \mathbb{Z}, p \in P)$, of $Q_{N,h}$ is of the form

$$g_{N,h}(k) = \int_{\Omega_h} \chi(\omega)dQ_{N,h} = \frac{1}{V_N} \sum_{k=1}^{N} w(k) \prod_{p \in P \setminus P_0}^* p^{-ikk_p h} \prod_{p \in P_0} p^{-ikh(k_p + l\alpha_p)}, \quad l \in \mathbb{Z}. \quad (2.11)$$

If $k_p = 0$ for all $p \in P \setminus P_0$ and $k_p = r\alpha_p$ for all $p \in P_0$ with some $r \in \mathbb{Z}$ (case 1), then

$$g_{N,h}(k) = \frac{1}{V_N} \sum_{k=1}^{N} w(k)\omega_d\prod_{p \in P_0} \omega^{d\alpha_p} = 1 \quad (2.12)$$

because $\prod_{p \in P_0} \omega^{d\alpha_p} = 1$ with $d \in \mathbb{Z}$.

Now, suppose that $k_p \neq 0$ for some $p \in P \setminus P_0$, or there does not exist $r \in \mathbb{Z}$ such that $k_p = r\alpha_p$ for all $p \in P_0$ (case 2). In [14], it was obtained that

$$\exp\{-ihA_p(k_p, l\alpha_p)\} \neq 1,$$

where

$$A_p(k_p, l\alpha_p) = \sum_{p \in P \setminus P_0}^* k_p \log p + \sum_{p \in P_0} (k_p + l\alpha_p) \log p, \quad l \in \mathbb{Z}.$$

Hence, we find that, for $u \geq 1$,

$$\sum_{k \leq u} \exp\{-ikhA_p(k_p, l\alpha_p)\} = \exp\{-ihA_p(k_p, l\alpha_p)\} - \exp\{-ih([u] + 1)A_p(k_p, l\alpha_p)\} \equiv \Sigma_h(u).$$

Therefore, in view of (2.11),

$$g_{N,h}(k) = \frac{w(N)\Sigma_h(N)}{V_N} - \frac{1}{V_N} \int_1^N \Sigma_h(u)w'(u)du.$$

Using the properties of the function $w$, hence we find that

$$g_{N,h}(k) = 0.$$

This together with (2.12) shows that

$$\lim_{N \to \infty} g_{N,h}(k) = \begin{cases} 1, & \text{in the case 1}, \\ 0, & \text{in the case 2}. \end{cases}$$

Since the right-hand side of the equality is the Fourier transform of the Haar measure $m_h^H$, the lemma follows by a continuity theorem for probability measures on compact groups. \(\Box\)

Now, together with $P_{N,n,h}$, consider

$$\hat{P}_{N,n,h}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k)I_{k : \zeta_{n,h}(s + ikh, \omega) \in A}(k), \quad A \in \mathcal{B}(H(D)),$$

with $\omega \in \Omega_h$. 
Lemma 5. Suppose that \( w(t) \in W \) and \( h \) is of type 2. Then \( P_{N,n,h} \) and \( \hat{P}_{N,n,h} \) both converge weakly to the measure \( m_H^h u_{n,h}^{-1} \) as \( N \to \infty \), where \( u_{n,h} : \Omega_h \to H(D) \) is given by \( u_{n,h}(\omega) = \zeta_n(s,\omega), \) \( \omega \in \Omega_h \).

Proof. By proving Lemma 2, in view of Lemma 4, we have that \( P_{N,n,h} \) converges weakly to \( m_H^h u_{n,h}^{-1} \) as \( N \to \infty \). Similarly, we obtain that if \( \hat{u}_{n,h}(\omega) : \Omega_h \to H(D) \) is given by

\[
\hat{u}_{n,h}(\omega) = \zeta_n(s,\omega\omega), \quad \omega \in \Omega_h,
\]
then \( \hat{P}_{N,n,h} \) converges weakly to \( m_H^h \hat{u}_{n,h}^{-1} \). However, \( \hat{u}_{n,h} = u_{n,h}(u) \), where \( u : \Omega_h \to \Omega_h \) is given by \( u(\omega) = \omega\omega \). This and the invariance of the Haar measure \( m_H^h \) show that \( m_H^h \hat{u}_{n,h}^{-1} = m_H^h u_{n,h}^{-1}. \)

For further considerations, we need some elements of the ergodic theory. Let \( a_h = (p^{-ih} : p \in \mathbb{P}) \). Then \( a_h \) is an element of \( \Omega_h \). Define the transformation \( \varphi_h(\omega) \) of \( \Omega_h \) by

\[
\varphi_h(\omega) = a_h \omega, \quad \omega \in \Omega_h.
\]
Then we have that \( \varphi_h \) is a measurable measure preserving transformation on the probability space \( (\Omega_h, \mathcal{B}(\Omega_h), m_H^h) \). We recall that a set \( A \in \mathcal{B}(\Omega_h) \) is called invariant with respect to \( \varphi_h \) if the sets \( A \) and \( \varphi_h(A) \) can differ from each other at most by a set of \( m_H^h \)-measure zero. The transformation \( \varphi_h \) is called ergodic if the \( \sigma \)-field of invariant sets of \( \Omega_h \) consists only of the sets having \( m_H^h \)-measure 1 or 0.

Lemma 6. Suppose that \( h \) is of type 2. Then the transformation \( \varphi_h \) is ergodic.

Proof of the lemma is given in [14, Lemma 3]. Let, for \( \omega \in \Omega_h \),

\[
\zeta_h(s,\omega) = \prod_p \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1}.
\]
The first application of Lemma 6 is devoted to the discrete mean square of \( \zeta_h(s,\omega) \).

Lemma 7. Suppose that \( w(t) \in W, h > 0 \) is of type 2, \( \sigma, \frac{1}{2} < \sigma < 1 \), is fixed and \( t \in \mathbb{R} \). Then, for almost all \( \omega \in \Omega_h \),

\[
\sum_{k=1}^N w(k) |\zeta_h(\sigma + it + ikh,\omega)|^2 \ll V_N(1 + |t|).
\]

Proof. We have that \( \zeta_h(s,\omega) \) coincides with the restriction of the random element \( \zeta(s,\omega) \) to the space \( (\Omega_h, \mathcal{B}(\Omega_h), m_H^h) \). First we consider the expectation \( \mathbb{E}[\zeta_h(\sigma + it,\omega)]^2 \). We write \( \zeta_h(s,\omega) \) in the form

\[
\zeta_h(\sigma + it,\omega) = \prod_{p \in \mathbb{P}_0} \left( 1 - \frac{\omega(p)}{p^s + it} \right)^{-1} \prod_{p \in \mathbb{P} \setminus \mathbb{P}_0} \left( 1 - \frac{\omega(p)}{p^s + it} \right)^{-1} \text{def} X_1 X_2.
\]
The random elements $X_1$ and $X_2$ are independent, moreover, for almost all $\omega \in \Omega_h$,

$$X_2 = \sum' \frac{\omega(m)}{m^{\sigma+it}},$$

where the sign $'$ means that the summing runs over $m = 1$ and $m \in \mathbb{N}$ with the canonical representation consisting only of primes $p \in \mathbb{P} \setminus \mathbb{P}_0$. In the series for $X_2$, the random variables are orthogonal, therefore,

$$\mathbb{E}|X_2|^2 = \sum' \frac{1}{m^{2\sigma}} < \infty.$$  

Clearly, $\mathbb{E}|X_1|^2$ is bounded by a constant. Therefore, there exists a finite constant $c > 0$ such that, for $\frac{1}{2} < \sigma < 1$ and $t \in \mathbb{R}$,

$$\mathbb{E} |\zeta_h(\sigma + it, \omega)|^2 = \mathbb{E}|X_1|^2 \mathbb{E}|X_2|^2 \leq c.$$  

Then (2.13), Lemma 6, the Birkhoff-Khintchine ergodic theorem, see, for example, [23], and the definition of the transformation $\varphi_h$ show that, for $\frac{1}{2} < \sigma < 1$ and $|t_0| < h$,

$$\sum_{k=1}^{N} |\zeta_h(\sigma + it + ikh, \omega)|^2 = \sum_{k=1}^{N} |\zeta_h(\sigma + it_0, \varphi^k_h(\omega))|^2$$

$$= N \mathbb{E} |\zeta_h(\sigma + it_0, \omega)|^2 (1 + o(1)) \ll N$$

for almost all $\omega \in \Omega_h$ as $N \to \infty$. Hence, denoting by $[u]$ the integer part of $u \in \mathbb{R}$, for $\frac{1}{2} < \sigma < 1$ and $|t| \leq h$, we find that

$$\sum_{k=1}^{N} |\zeta_h(\sigma + it + ikh, \omega)|^2 = \sum_{k=1+[t/h]}^{N+[t/h]} |\zeta_h(\sigma + it + ikh, \omega)|^2 \ll N (1 + |t|)$$

for almost all $\omega \in \Omega_h$. From this, summing by parts, we obtain the estimate of the lemma.

Similarly to the proof of Lemma 3, we arrive, by using Lemma 7, to

**Lemma 8.** Suppose that $w(t) \in W$ and $h > 0$ is of type 2. Then, for almost all $\omega \in \Omega_h$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) \rho(\zeta_h(s + ikh, \omega), \zeta_{n,h}(s + ikh, \omega)) = 0.$$  

For $\omega \in \Omega_h$, additionally to the measure $P_{N,h}$, define

$$\hat{P}_{N,h}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\{k : \zeta_h(s + ikh, \omega) \in A\}}(k), \quad A \in \mathcal{B}(H(D)).$$  

Then, using Lemmas 3, 5 and 8, and repeating the first part of the proof of Theorem 6, we obtain
Lemma 9. Suppose that \( w(t) \in W \) and \( h > 0 \) is of type 2. Then, on \( (H(D), \mathcal{B}(H(D))) \), there exists a probability measure \( P_h \) such that \( P_{N,h} \) and \( \hat{P}_{N,h} \) both converge weakly to \( P_h \) as \( N \to \infty \).

Denote by \( P_{\zeta,h} \) the distribution of the random element \( \zeta_h(s,\omega), \omega \in \Omega_h \). Then we have the following analogue of Theorem 6.

Theorem 7. Suppose that \( w(t) \in W \) and \( h > 0 \) is of type 2. Then \( P_{N,h} \) converges weakly \( P_{\zeta,h} \) as \( N \to \infty \). Moreover, the support of the measure \( P_{\zeta,h} \) is the set \( S \).

Proof. In virtue of Lemma 9, it suffices to identify the measure \( P \) in that lemma, and to find the support of the limit measure. For the first problem, we will apply Lemma 6, and the Birkhoff-Khintchine theorem. Let \( A \) be a continuity set of \( P \). On the probability space \((\Omega_h, \mathcal{B}(\Omega_h), m_H)\), define the random variable \( \xi \) by the formula

\[
\xi(\omega) = \begin{cases} 
1, & \text{if } \zeta_h(s,\omega) \in A, \\
0, & \text{otherwise}.
\end{cases}
\]

Then we have that

\[
E\xi = \int_{\Omega_h} \xi(\omega) dm_H = P_{\zeta,h}(A). \tag{2.14}
\]

Moreover, by Lemma 9,

\[
\lim_{N \to \infty} \hat{P}_N(A) = P_h(A). \tag{2.15}
\]

In view of Lemma 6 and the Birkhoff-Khintchine theorem, for almost all \( \omega \in \Omega_h \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \xi(\varphi^k_h(\omega)) = E\xi.
\]

Since \( w \in W \), from this it follows that, for almost all \( \omega \in \Omega_h \),

\[
\lim_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) \xi(\varphi^k_h(\omega)) = E\xi. \tag{2.16}
\]

However, by the definition of \( \varphi_h \),

\[
\frac{1}{V_N} \sum_{k=1}^{N} w(k) \xi(\varphi^k_h(\omega)) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\{k; \zeta_h(s+ikh,\omega) \in A\}}(k) = \hat{P}_{N,h}(A).
\]

Therefore, by (2.14) and (2.16),

\[
\lim_{N \to \infty} \hat{P}_{N,h}(A) = P_{\zeta,h}(A).
\]

This and (2.15) show that \( P_h = P_{\zeta,h} \).

For finding the support of \( P_{\zeta,h} \), we use the representation (2.13). For \( p \in \mathbb{P} \setminus \mathbb{P}_0 \), the random variables \( \omega(p) \) are independent. Thus, by the proof of Lemma 6.5.5 from [11], we find that the support of the random element \( X_2 \) is the set \( S \). Since the random elements \( X_1 \) and \( X_2 \) are independent and \( X_1 \) is not degenerate at zero, we obtain that the support of \( X_1X_2 \) is the set \( S \), i.e., the support of the measure \( P_{\zeta,h} \) is the set \( S \). The theorem is proved. □
3 Proof of universality theorems

Theorems 4 and 5 follow from the limit theorems (Theorems 6 and 7), for \( \zeta(s) \) as well as from the Mergelyan theorem [19] on the approximation of analytic functions by polynomials.

**Proof.** (Of Theorem 4). By the Mergelyan theorem, there exists a polynomial \( p(s) \) such that
\[
\sup_{s \in K} \left| f(s) - e^{p(s)} \right| < \frac{\varepsilon}{2}. \tag{3.1}
\]

For brevity, denote the limit measure in Theorems 6 and 7 by \( \hat{P}_\zeta \), i.e.,
\[
\hat{P}_\zeta = \begin{cases} 
P_\zeta, & \text{if } h \text{ is of type 1,} \\
P_{\zeta,h}, & \text{if } h \text{ is of type 2,}
\end{cases}
\]
\[
\hat{P}_N = \begin{cases} 
P_N, & \text{if } h \text{ is of type 1,} \\
P_{N,h}, & \text{if } h \text{ is of type 2.}
\end{cases}
\]

Then we have that \( \hat{P}_N \) converges weakly to \( \hat{P}_\zeta \) as \( N \to \infty \). Define the set
\[
G_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} \left| g(s) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\}.
\]

Since \( e^{p(s)} \neq 0 \), and, in view of Theorems 6 and 7, the support of the measure \( \hat{P}_\zeta \) is the set \( S \), the set \( G_\varepsilon \) is an open neighbourhood of an element of the support, therefore,
\[
\hat{P}_\zeta(G_\varepsilon) > 0. \tag{3.2}
\]

Moreover, by the first parts of Theorems 6 and 7, and the equivalent of weak convergence of probability measures in terms of open sets [2, Theorem 2.1], we have that
\[
\liminf_{N \to \infty} \hat{P}_N(G_\varepsilon) \geq \hat{P}_\zeta(G_\varepsilon).
\]

This, (3.2) and the definitions of \( \hat{P}_N \) and \( G_\varepsilon \) show that
\[
\liminf_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I \left\{ k : \sup_{s \in K} |\zeta(s + ikh) - e^{p(s)}| < \frac{\varepsilon}{2} \right\} (k) > 0. \tag{3.3}
\]

It remains to replace \( e^{p(s)} \) by \( f(s) \) in the latter inequality. Suppose that \( k \) satisfies the inequality
\[
\sup_{s \in K} \left| \zeta(s + ikh) - e^{p(s)} \right| < \frac{\varepsilon}{2}.
\]

Then, in virtue of (3.1), the same \( k \) satisfies the inequality
\[
\sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon.
\]

Therefore,
\[
\left\{ k : \sup_{s \in K} |\zeta(s + ikh) - e^{p(s)}| < \frac{\varepsilon}{2} \right\} \subset \left\{ k : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\}.
\]
This inclusion together with (3.3) proves the theorem. □

**Proof.** (Of Theorem 5). Define the set

\[ \hat{G}_\varepsilon = \{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \}. \]

Then \( \partial \hat{G}_\varepsilon = \{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \} \) is the boundary of \( \hat{G}_\varepsilon \).

Hence, \( \partial \hat{G}_{\varepsilon_1} \cap \partial \hat{G}_{\varepsilon_2} = \emptyset \) if \( \varepsilon_1 \neq \varepsilon_2, \varepsilon_1 > 0, \varepsilon_2 > 0 \). Therefore, the set \( \partial \hat{G}_\varepsilon \) can have a positive \( \hat{P}_\zeta \)-measure for at most countably many \( \varepsilon > 0 \). This means that the set \( \hat{G}_\varepsilon \) is a continuity set of the measure \( \hat{P}_\zeta \) for all but at most countably many \( \varepsilon > 0 \). Using Theorems 6 and 7, and the equivalent of weak convergence of probability measures in terms of continuity sets [2, Theorem 2.1], we have that

\[ \lim_{N \to \infty} \hat{P}_N(\hat{G}_\varepsilon) = \hat{P}_\zeta(\hat{G}_\varepsilon) \] (3.4)

for all but at most countably many \( \varepsilon > 0 \). Moreover, (3.1) shows that \( G_\varepsilon \subset \hat{G}_\varepsilon \).

Therefore, by (3.2), \( \hat{P}_\zeta(\hat{G}_\varepsilon) > 0 \). This, (3.4) and the definition of the set \( \hat{G}_\varepsilon \) prove the theorem. □

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A. Laurinčikas, D. Šiaučiūnas and G. Vadeikis


