A Weighted Version of the Mishou Theorem

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Received April 9, 2020; revised September 27, 2020; accepted September 27, 2020

Abstract. In 2007, H. Mishou obtained a joint universality theorem for the Riemann and Hurwitz zeta-functions $\zeta(s)$ and $\zeta(s,\alpha)$ with transcendental parameter $\alpha$ on the approximation of a pair of analytic functions by shifts $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$, $\tau \in \mathbb{R}$. In the paper, the Mishou theorem is generalized for the set of above shifts having a weighted positive lower density. Also, the case of a positive density is considered.

Keywords: Hurwitz zeta-function, Mishou theorem, Riemann zeta-function, universality.

AMS Subject Classification: 11M06; 11M41.

1 Introduction

The Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, and the Hurwitz zeta-function $\zeta(s,\alpha)$ with parameter $0 < \alpha \leq 1$ are defined, for $\sigma > 1$, by the Dirichlet series

$$
\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \quad \text{and} \quad \zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},
$$

and have analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The functions $\zeta(s)$ and $\zeta(s,\alpha)$ play an important role not only in analytic number theory but in mathematics in...
general. The definitions of $\zeta(s)$ and $\zeta(s, \alpha)$ are similar, however, their analytic properties are quite different. For example, since the function $\zeta(s)$, for $\sigma > 1$, has the Euler product over primes

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

$\zeta(s) \neq 0$ in the half-plane $\sigma > 1$, while the function $\zeta(s, \alpha)$ has zeros in that half-plane if $\alpha \neq 1$ or $1/2$. On the other hand, the functions $\zeta(s)$ and $\zeta(s, \alpha)$ have a common feature, they are universal in the sense that their shifts $\zeta(s + i\tau)$ and $\zeta(s + i\tau, \alpha)$ approximate wide classes of analytic functions. We recall that universality of the function $\zeta(s)$ was discovered by S.M. Voronin in [31]. For a statement of the Voronin theorem, it is convenient to use the following notation. For $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, denote by $\mathcal{K}$ the class of compact subsets of the strip $D$ with connected complements, by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on $K$ that are analytic in the interior of $K$, and by $H_0(K)$ the subclass of $H(K)$ of non-vanishing functions on $K$. Then the modern version of the Voronin theorem, see, for example, [1, 6, 13, 30] asserts that, for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$, and $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

The latter inequality shows that there are infinitely many shifts $\zeta(s + i\tau)$ approximating a given function $f(s) \in H_0(K)$. Here $\text{meas}A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Universality of the Hurwitz zeta-function is a more complicated problem. At the moment, the following result is known. Suppose that $\alpha$ is a transcendental or rational $\neq 1, 1/2$. Then, for every $K \in \mathcal{K}$, $f(s) \in H(K)$, and $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

The case of rational $\alpha$ was obtained by Voronin [32] and B. Bagchi [1], while the case of transcendental $\alpha$ was treated by S.M. Gonek [6], and, by a different method, in [23]. In [14], the transcendence of $\alpha$ was replaced by a weaker condition on the linear independence of the set $L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ over the field of rational numbers $\mathbb{Q}$.

H. Mishou in [29] began to study a joint approximation property of the functions $\zeta(s)$ and $\zeta(s, \alpha)$. More precisely, he proved that if $\alpha$ is transcendental, then, for every $K_1, K_2 \in \mathcal{K}$, $f_1(s) \in H_0(K_1)$, $f_2 \in H(K_2)$ and $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f(s)| < \varepsilon, \right.$$

$$\sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \left.\right\} > 0.$$

The Mishou theorem is the first so-called mixed joint universality theorem because the function $\zeta(s)$ has Euler’s product over primes, while the function
\(\zeta(s, \alpha)\) with transcendental \(\alpha\) has no such a product. Mixed joint universality theorems were studied in [2, 5, 7, 8, 9, 10, 11, 15, 16, 17, 18, 19, 20, 21, 22, 24].

The aim of this paper, is a joint weighted universality theorem for the functions \(\zeta(s)\) and \(\zeta(s, \alpha)\). The weighted universality of zeta-functions was began to study in [12]. In weighted universality theorems, the positivity of a lower density of the shifts approximating a given analytic function is replaced by the positivity of that weighted analogue. Let \(w(\tau)\) be positive function for \(\tau \geq T_0 > 0\) such that

\[
\lim_{T \to \infty} W(T, w) = \infty, \quad W(T, w) = \int_{T_0}^T w(\tau) \, d\tau,
\]

and, for every interval \([a, b] \subset [T_0, \infty)\), the variation \(V^b_a w\) satisfies the inequality \(V^b_a w \leq cw(a)\) with certain \(c > 0\). Moreover, let \(I(A)\) denote the indicator function of the set \(A\). Under the above hypotheses on the weight function \(w\), it was obtained in [12] that, for every \(K \in \mathcal{K}, f(s) \in H_0(K)\), and \(\varepsilon > 0\),

\[
\liminf_{T \to \infty} \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) I \left( \left\{ \tau \in [T_0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} \right) \, d\tau > 0.
\]

A weighted discrete universality for \(\zeta(s)\) was proved in [25]. Weighted universality theorems for periodic zeta-functions were obtained in [26, 27].

A weighted universality theorem for the Hurwitz zeta-function was proved in [3]. Denote by \(W\) the above class of weight functions.

**Theorem 1.** Suppose that \(\alpha\) is transcendental and \(w \in W\). Let \(K \in \mathcal{K}\) and \(f(s) \in H(K)\). Then, for every \(\varepsilon > 0\),

\[
\liminf_{T \to \infty} \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) I \left( \left\{ \tau \in [T_0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} \right) \, d\tau > 0.
\]

The main result of this paper is the following weighted theorem.

**Theorem 2.** Suppose that \(\alpha\) is transcendental and \(w \in W\). Let \(K_1, K_2 \in \mathcal{K}\) and \(f(s) \in H_0(K_1), f_2(s) \in H(K_2)\). Then, for every \(\varepsilon > 0\),

\[
\liminf_{T \to \infty} \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) I \left( \left\{ \tau \in [T_0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} \right) \, d\tau > 0.
\]

Moreover, the limit

\[
\lim_{T \to \infty} \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) I \left( \left\{ \tau \in [T_0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} \right) \, d\tau > 0
\]

exists for all but at most countably many \(\varepsilon > 0\).

If \(w(\tau) = 1\), then the first assertion of Theorem 2 reduces to the Mishou theorem [29]. For example, we may take \(w(\tau) = 1/\tau\) and \(\alpha = 1/e\).

For the proof of Theorem 2, we will use the probabilistic approach based on weak convergence of probability measures in the space of analytic functions.
2 A weighted limit theorem on the product of two tori

In what follows, we denote by $B(\mathbb{X})$ the Borel $\sigma$-field of the space $\mathbb{X}$, by $\mathbb{P}$ the set of all prime numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. Define two tori $\Omega_1 = \prod_{p \in \mathbb{P}} \gamma_p$ and $\Omega_2 = \prod_{m \in \mathbb{N}_0} \gamma_m$, where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. With product topology and pointwise multiplication, the infinite-dimensional tori $\Omega_1$ and $\Omega_2$ are compact topological Abelian groups. Therefore, $\Omega = \Omega_1 \times \Omega_2$ is again a compact topological Abelian group. Hence, on $(\Omega, B(\Omega))$, the probability Haar measure $m_H$ can be defined, and we obtain the probability space $(\Omega, B(\Omega), m_H)$. Denote by $\omega_1(p)$ the $p$th component of an element $\omega_1 \in \Omega_1$, $p \in \mathbb{P}$, and by $\omega_2(m)$ the $m$th component of an element $\omega_2 \in \Omega_2$, $m \in \mathbb{N}_0$. The elements of $\Omega$ are of the form $\omega = (\omega_1, \omega_2)$.

In this section, we will consider the weak convergence for

$$Q_{T, w}(A) = \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) I \left( \{ \tau \in [T_0, T] : (p^{-i\tau} : p \in \mathbb{P}), (m + \alpha)^{-i\tau} : m \in \mathbb{N}_0 \} \in A \} \right) d\tau, \quad A \in B(\Omega).$$

**Theorem 3.** Suppose that $\alpha$ is transcendental and $w \in W$. Then $Q_{T, w}$ converges weakly to the Haar measure $m_H$ as $T \to \infty$.

**Proof.** The characters of the group $\Omega$ are of the form

$$\prod_{p \in \mathbb{P}} \omega_1^{k_p}(p) \prod_{m \in \mathbb{N}_0} \omega_2^{l_m}(m),$$

where the sign "'" means that only a finite number of integers $k_p$ and $l_m$ are distinct from zero. Therefore, the Fourier transform $g_{T, w}(\mathbf{k}, \mathbf{l})$, $\mathbf{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, $\mathbf{l} = (k_p : l_m \in \mathbb{Z}, m \in \mathbb{N}_0)$, of $Q_{T, w}$ is defined by

$$g_{T, w}(\mathbf{k}, \mathbf{l}) = \int_{\Omega} \prod_{p \in \mathbb{P}} \omega_1^{k_p}(p) \prod_{m \in \mathbb{N}_0} \omega_2^{l_m}(m) dQ_{T, w}.$$ 

Therefore, by the definition of $Q_{T, w}$,

$$g_{T, w}(\mathbf{k}, \mathbf{l}) = \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) \prod_{p \in \mathbb{P}} p^{-ik_p\tau} \prod_{m \in \mathbb{N}_0} (m + \alpha)^{-il_m\tau} d\tau$$

$$= \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) \exp \left\{-i\tau \left( \sum_{p \in \mathbb{P}} k_p \log p + \sum_{m \in \mathbb{N}_0} l_m \log (m+\alpha) \right) \right\} d\tau. \quad (2.1)$$

Clearly,

$$g_{T, w}(0, 0) = \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) d\tau = 1. \quad (2.2)$$
Suppose that \((k, l) \neq (0, 0)\). Then
\[
A(k, l) \overset{\text{def}}{=} \sum_{p \in \mathbb{P}} k_p \log p + \sum_{m \in \mathbb{N}_0} l_m \log(m + \alpha) \neq 0. \tag{2.3}
\]

Actually, if the latter inequality is not true, then
\[
\prod_{p \in \mathbb{P}} p^{k_p} \prod_{m \in \mathbb{N}_0} (m + \alpha)^{l_m} = 1.
\]

From this, it follows that
\[
\prod_{m \in \mathbb{N}_0} (m + \alpha)^{l_m}
\]
is a rational number. However, this contradicts the transcendence of \(\alpha\). If all \(l_m = 0\), then \(\sum_{p \in \mathbb{P}} k_p \log p \neq 0\) because the set \(\{\log p : p \in \mathbb{P}\}\) is linearly independent over the field of rational numbers. Thus, (2.3) is true. Now, by (2.1), we find
\[
g_{T, w}(k, l) = \frac{1}{-iW(T, w)A(k, l)} \int_{T_0}^{T} w(\tau) \exp\{-i\tau A(k, l)\}
\]
\[
\ll (W(T, w)|A(k, l)|)^{-1} \left(1 + \int_{T_0}^{T} |dw(\tau)|\right) \ll (W(T, w)|A(k, l)|)^{-1}
\]
in view of a property of the variation of \(w(\tau)\). Since \(\lim_{T \to \infty} W(T, w) = \infty\), this shows that
\[
\lim_{T \to \infty} g_{T, w}(k, l) = 0.
\]

Therefore, by (2.2),
\[
\lim_{T \to \infty} g_{T, w}(k, l) = \begin{cases} 
1 & \text{if } (k, l) = (0, 0), \\
0 & \text{if } (k, l) \neq (0, 0),
\end{cases}
\]
and the theorem is proved because the right-hand side of the latter equality is the Fourier transform of the Haar measure \(m_H\).

### 3 Case of absolute convergence

Theorem 3 implies a weighted joint limit theorem in the space \(H^2(D)\), where \(H(D)\) is the space of analytic functions on \(D\) endowed with the topology of uniform convergence on compacta. Thus, let \(\theta > 1/2\) be a fixed number, for \(m, n \in \mathbb{N}\),
\[
v_n(m) = \exp \left\{ - \left( \frac{m}{n} \right)^\theta \right\},
\]
and, for \(m \in \mathbb{N}_0, n \in \mathbb{N}\),
\[
v_n(m, \alpha) = \exp \left\{ - \left( \frac{m + \alpha}{n + \alpha} \right)^\theta \right\}.
\]
Define the series 
\[ \zeta_n(s) = \sum_{m=1}^{\infty} \frac{\nu_n(m)}{m^s} \quad \text{and} \quad \zeta_n(s,\alpha) = \sum_{m=0}^{\infty} \frac{\nu_n(m,\alpha)}{(m+\alpha)^s}, \]
Then the latter series are absolutely convergent for \( \sigma > 1/2 \), see [13, 23], respectively. For brevity, let 
\[ \zeta_n(s,\alpha) = (\zeta_n(s),\zeta_n(s,\alpha)). \]
Extend the functions \( \omega_1(p) \), to the set \( \mathbb{N} \) by the formula 
\[ \omega_1(m) = \prod_{p^l|m, p^{l+1} \nmid m} \omega_1(p), \quad m \in \mathbb{N}, \]
and, additionally to \( \zeta_n(s) \) and \( \zeta_n(s,\alpha) \), define 
\[ \zeta_n(s,\omega_1) = \sum_{m=1}^{\infty} \frac{\omega_1(m)\nu_n(m)}{m^s} \quad \text{and} \quad \zeta_n(s,\omega_2,\alpha) = \sum_{m=0}^{\infty} \frac{\omega_2(m)\nu_n(m,\alpha)}{(m+\alpha)^s}, \]
and put \( \zeta_n(s,\omega,\alpha) = (\zeta_n(s,\omega_1),\zeta_n(s,\omega_2,\alpha)) \). Obviously, the series \( \zeta_n(s,\omega_1) \) and \( \zeta_n(s,\omega_2,\alpha) \) are absolutely convergent for \( \sigma > 1/2 \) as well.

Consider the function \( u_n : \Omega \rightarrow H^2(D) \) given by \( u_n(\omega) = \zeta_n(s,\omega,\alpha) \). Since the above series are absolutely convergent for \( \sigma > 1/2 \), the function \( u_n(\omega) \) is continuous. For \( A \in \mathcal{B}(H^2(D)) \), define
\[ P_{T,n,w}(A) = \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I \left( \{ \tau \in [T_0,T] : \zeta_n(s+i\tau,\alpha) \in A \} \right) d\tau. \]
Then we have \( P_{T,n,w}(A) = Q_{T,w}(u^{-1}A) \). Thus, the equality \( P_{T,n,w} = Q_{T,w}u^{-1} \) is true. This, the continuity of \( u_n \), Theorem 3 together with Theorem 5.1 of [4] lead to the following theorem.

**Theorem 4.** Suppose that \( \alpha \) is transcendental and \( w \in W \). Then \( P_{T,n,w} \) converges weakly to the measure \( V_n \) as \( T \rightarrow \infty \).

The measure \( V_n \) plays an important role in the proof of the limit theorem for 
\[ P_{T,w}(A) = \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) I \left( \{ \tau \in [T_0,T] : \zeta(s+i\tau,\alpha) \in A \} \right) d\tau, \]
where \( \zeta(s,\alpha) = (\zeta(s),\zeta(s,\alpha)) \). From the proof of the Mishou theorem [29], the following properties of \( V_n \) follows. On the probability space \( (\Omega,\mathcal{B}(\Omega),m_H) \), define the \( H^2(D) \)-valued random element 
\[ \zeta(s,\omega,\alpha) = \left( \prod_{p \in \mathbb{P}} \left( 1 - \frac{\omega_1(p)}{p^s} \right)^{-1}, \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^s} \right), \]
and let $P_\zeta$ be the distribution of $\zeta(s, \omega, \alpha)$, i.e.,

$$P_\zeta(A) = m_H \{ \omega \in \Omega : \zeta(s, \omega, \alpha) \in A \}, \quad A \in \mathcal{B}(H^2(D)).$$

Moreover, let $S = \{ g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}$. Under the above notation, we have

**Lemma 1.** Suppose that $\alpha$ is transcendental. Then $V_n$ converges weakly to $P_\zeta$ as $n \to \infty$. Moreover, the support of $P_\zeta$ is the set $S \times H(D)$.

To prove that $P_{T,w}$, as $T \to \infty$, also converges weakly to the measure $P_\zeta$, some approximation of $\zeta(s, \alpha)$ by $\zeta_n(s, \alpha)$ is needed.

### 4 Approximation in the mean

For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact subsets such that $D = \bigcup_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K$ lies in some $K_l$. Then $\rho$ is a metric on $H(D)$ that induces its topology of uniform convergence on compacta.

Now, let $g_1 = (g_{11}, g_{12})$, $g_2 = (g_{21}, g_{22}) \in H^2(D)$. Then putting

$$\rho(g_1, g_2) = \max_{1 \leq j \leq 2} \rho(g_{1j}, g_{2j})$$

gives a metric on $H^2(D)$ inducing the product topology.

The following statement is true.

**Theorem 5.** Suppose that $w \in W$. Then

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \rho \left( \zeta(s + i\tau, \alpha), \zeta_n(s + i\tau, \alpha) \right) d\tau = 0$$

for all $0 < \alpha \leq 1$.

**Proof.** By the definition of the metric $\rho$, it suffices to prove the equalities

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \rho \left( \zeta(s + i\tau, \alpha), \zeta_n(s + i\tau) \right) d\tau = 0 \quad (4.1)$$

and

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \rho \left( \zeta(s + i\tau, \alpha), \zeta_n(s + i\tau, \alpha) \right) d\tau = 0. \quad (4.2)$$

Obviously, (4.1) is a corollary of (4.2) with $\alpha = 1$. Moreover, to prove (4.2) it suffices to show that, for every compact set $K \subset D$, 
\[
\lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) \sup_{s \in K} |\zeta(s + i\tau, \alpha) - \zeta_n(s + i\tau, \alpha)| \, d\tau = 0. \quad (4.3)
\]

Let 
\[
l_n(s, \alpha) = \frac{s}{\theta} \Gamma \left( \frac{s}{\theta} \right) (n + \alpha)^s, \quad n \in \mathbb{N},
\]
where $\Gamma(s)$ is the Euler gamma-function. Then the classical Mellin formula implies, for $\sigma > 1/2$, the equality 
\[
\zeta_n(s, \alpha) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s + z, \alpha) l_n(z, \alpha) \frac{dz}{z}. \quad (4.4)
\]

We take an arbitrary compact set $K \subset D$, and fix $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon$ for points $s = \sigma + iv \in K$. Then, by (4.4) and the residue theorem, for $\theta_1 > 0$, 
\[
\zeta_n(s, \alpha) - \zeta(s, \alpha) = \frac{1}{2\pi i} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \zeta(s + z, \alpha) l_n(z, \alpha) \frac{dz}{z} + R_n(s, \alpha), \quad (4.5)
\]
where $R_n(s, \alpha) = l_n(1 - s, \alpha)/(1 - s)$. Suppose that $\theta_1 = \sigma - \varepsilon - 1/2$. Then (4.5) shows that, for $s \in K$, 
\[
|\zeta_n(s, \alpha) - \zeta(s, \alpha)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\zeta(s + it - \theta_1 + it, \alpha)| \frac{|l_n(-\theta_1 + it, \alpha)|}{|\theta_1 + it|} \, dt \nonumber
\]
\[
+ |R_n(s + it, \alpha)|. 
\]

Hence, after shifting $v + t$ to $t$, we obtain 
\[
\frac{1}{W(T, w)} \int_{T_0}^T w(\tau) \sup_{s \in K} |\zeta(s + i\tau, \alpha) - \zeta_n(s + i\tau, \alpha)| \, d\tau \ll I_1 + I_2, \quad (4.6)
\]
where 
\[
I_1 = \int_{-\infty}^{\infty} w(\tau) \left( \frac{1}{W(T, w)} \int_{T_0}^T |\zeta(1/2 + \varepsilon + i(t + \tau, \alpha)| \, d\tau \right) \nonumber
\]
\[
\times \sup_{s \in K} \frac{|l_n(1/2 + \varepsilon - s + it, \alpha)|}{|1/2 + \varepsilon - s + it|} \, dt, \nonumber
\]
\[
I_2 = \frac{1}{W(T, w)} \int_{T_0}^T w(\tau) \sup_{s \in K} |R_n(s + i\tau, \alpha)| \, d\tau. \nonumber
\]

It is well known that $\Gamma(s + it) \ll \exp\{-c|t|\}$ uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$ for every $\sigma_1 < \sigma_2$ with an absolute constant $c > 0$. Therefore, putting $\theta = 1/2 + \varepsilon$, we find that, for $s \in K$, 
\[
\frac{|l_n(1/2 + \varepsilon - s + it, \alpha)|}{|1/2 + \varepsilon - s + it|} = \frac{(n + \alpha)^{1/2+\varepsilon-\sigma}}{\theta} \left| \Gamma \left( \frac{1/2 + \varepsilon - \sigma}{\theta} + \frac{i(t - v)}{\theta} \right) \right| \nonumber
\]
\[
\ll \frac{n^{-\varepsilon}}{\theta} \exp \left\{ -c \left| \frac{t - v}{\theta} \right| \right\} \ll_{K, \alpha} n^{-\varepsilon} \exp \{-c_1|t|\} \quad (4.7)
\]
with $c_1 > 0$. In [3] it was obtained that, for $\sigma$, \(1/2 < \sigma < 1\), and $t \in \mathbb{R}$,
\[
\int_{T_0}^{T} w(\tau) |\zeta(\sigma + i(t + \tau), \alpha)|^2 \, dt \ll W(t, w)(1 + |t|^2).
\]
Hence,
\[
\int_{T_0}^{T} w(\tau) |\zeta(\sigma + i(t + \tau), \alpha)|^2 \, d\tau \ll \left( \int_{T_0}^{T} w(\tau) \, d\tau \right)^{1/2} \ll W(t, w)(1 + |t|^2).
\]
This together with (4.7) shows that
\[
I_1 \ll_K n^{-\varepsilon} \int_{-\infty}^{\infty} (1 + |t|) \exp\{-c_1 |t|\} \, dt \ll_{K, \alpha} n^{-\varepsilon}. \quad (4.8)
\]
Similarly, we find that, for $s \in K$,
\[
|R_n(s + i\tau, \alpha)| \ll_{\alpha} n^{1-\sigma} \exp\left\{-c \frac{|\tau - v|}{\theta} \right\} \ll_{K, \alpha} n^{1-\sigma} \exp\{-c_2 |\tau|\}
\]
with $c_2 > 0$. Thus,
\[
I_2 \ll_{K, \alpha} n^{1/2-2\varepsilon} \frac{1}{W(T, w)} \int_{T_0}^{T} w(\tau) \exp\{-c_2 |\tau|\} \, d\tau \ll_{K, \alpha} n^{1/2-2\varepsilon} W(T, w).
\]
If $T \to \infty$, then $I_2 \to 0$, because $W(T, w) \to \infty$. Moreover, by (4.8), if $n \to \infty$, then $I_1 \to 0$. Therefore, (4.6) implies (4.3). The lemma is proved. \(\square\)

5 \ A limit theorem for $\zeta(s, \alpha)$

Now we are ready to prove the weak convergence for $P_{T,w}$ as $T \to \infty$.

**Theorem 6.** Suppose that $\alpha$ is transcendental and $w \in W$. Then $P_{T,w}$ converges weakly to the measure $P_{\zeta}$ as $T \to \infty$.

**Proof.** On a certain probability space with measure $\mu$, define a random variable $\theta_{T,w}$ by
\[
\mu\{\theta_{T,w} \in A\} = \frac{1}{W(T, w)} \int_{T_0}^{T} w(\tau) I(A) \, d\tau, \quad A \in \mathcal{B}(\mathbb{R}).
\]
Consider the $H^2(D)$-valued random element
\[
X_{T,n,w} = X_{T,n,w}(s) = \zeta_n(s + i\theta_{T,w}, \alpha).
\]
Then, in view of Theorem 4,
\[
X_{T,n,w} \xrightarrow{D} Y_n, \quad (5.1)
\]
where $Y_n$ is the $H^2(D)$-valued random element with the distribution $V_n$. Lemma 1 implies the relation

$$Y_n \xrightarrow{D} P_{\zeta}.$$ 

Moreover, an application of Theorem 5 shows that, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \mu \left( p \left( X_{T,w}(s), X_{T,n,w}(s) \right) \geq \varepsilon \right) \ll \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\varepsilon W(T,w)} \int_{T_0}^{T} w(\tau) p \left( \zeta(s+i\tau, \alpha), \zeta_n(s+i\tau, \alpha) \right) d\tau = 0, \tag{5.2}$$

where the $H^2(D)$-valued random element $X_{T,w} = X_{T,w}(s)$ is defined by

$$X_{T,w}(s) = \zeta(s + i\theta_{T,w}, \alpha).$$

Now, relations (5.1)–(5.2) show that all hypotheses of Theorem 4.2 from [4] are satisfied. Therefore, we obtain that

$$X_{T,w} \xrightarrow{D} P_{\zeta},$$

and this is equivalent to the assertion of the theorem. $\Box$

6 Proof of universality

Theorem 2 follows easily from Theorem 6 and the Mergelyan theorem on the approximation of analytic functions by polynomials [28].

Proof. (Proof of Theorem 2). By the Mergelyan theorem, there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s \in K_1} \left| f_1(s) - e^{p_1(s)} \right| < \frac{\varepsilon}{2}, \quad \sup_{s \in K_2} \left| f_2(s) - p_2(s) \right| < \frac{\varepsilon}{2}. \tag{6.1}$$

Define the set

$$G_\varepsilon = \left\{ g_1, g_2 \in H(D) : \sup_{s \in K_1} \left| g_1(s) - e^{p_1(s)} \right| < \frac{\varepsilon}{2}, \sup_{s \in K_2} \left| g_2(s) - p_2(s) \right| < \frac{\varepsilon}{2} \right\}.$$

We observe that, in virtue of Lemma 1, $(e^{p_1(s)}, p_2(s))$ is an element of the support of the measure $P_{\zeta}$. Since $G_\varepsilon$ is an open neighbourhood of an element of the support of $P_{\zeta}$, the inequality

$$P_\zeta(G_\varepsilon) > 0 \tag{6.2}$$

is true. Therefore, using the equivalent of the weak convergence of probability measures in terms of open sets and taking into account Theorem 6, we have

$$\liminf_{T \to \infty} P_{T,w}(G_\varepsilon) \geq P_\zeta(G_\varepsilon) > 0.$$
Hence, by the definitions of $P_{T,w}$ and $G_\varepsilon$,

$$
\liminf_{T \to \infty} \frac{1}{W(T,w)} \int_{T_0}^{T} w(\tau) \left( \sup_{s \in K_1} \left| \zeta(s+i\tau) - \exp^{p_1(s)} \right| < \frac{\varepsilon}{2}, \sup_{s \in K_2} \left| \zeta(s+i\tau, \alpha) - p_2(s) \right| < \frac{\varepsilon}{2} \right) d\tau > 0.
$$

(6.3)

It remains to replace $\exp^{p_1(s)}$ and $p_2(s)$ by $f_1(s)$ and $f_2(s)$, respectively. Suppose that $\tau$ satisfy inequalities

$$
\sup_{s \in K_1} \left| \zeta(s+i\tau) - f_1(s) \right| < \varepsilon, \sup_{s \in K_2} \left| \zeta(s+i\tau, \alpha) - f_2(s) \right| < \varepsilon.
$$

Then inequalities (6.1) imply

$$
\sup_{s \in K_1} \left| \zeta(s+i\tau) - f_1(s) \right| < \varepsilon, \sup_{s \in K_2} \left| \zeta(s+i\tau, \alpha) - f_2(s) \right| < \varepsilon.
$$

Consequently,

$$
\left\{ \tau \in [T_0,T] : \sup_{s \in K_1} \left| \zeta(s+i\tau) - f_1(s) \right| < \varepsilon, \sup_{s \in K_2} \left| \zeta(s+i\tau, \alpha) - f_2(s) \right| < \varepsilon \right\}
\subset \left\{ \tau \in [T_0,T] : \sup_{s \in K_1} \left| \zeta(s+i\tau) - e^{p_1(s)} \right| < \frac{\varepsilon}{2}, \sup_{s \in K_2} \left| \zeta(s+i\tau, \alpha) - p_2(s) \right| < \frac{\varepsilon}{2} \right\}.
$$

This and (6.3) prove the first assertion of the theorem.

Define one more set

$$
\hat{G}_\varepsilon = \left\{ g_1, g_2 \in H(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |g_2(s) - f_2(s)| < \varepsilon \right\}.
$$

Then the boundaries \( \partial \hat{G}_{\varepsilon_1} \) and \( \partial \hat{G}_{\varepsilon_2} \) do not intersect for different positive \( \varepsilon_1 \) and \( \varepsilon_2 \). This shows that the set \( \hat{G}_\varepsilon \) is a continuity set of the measure \( P_{\zeta, \varepsilon} \) for all but at most countably many \( \varepsilon > 0 \). Therefore, using the equivalent of weak convergence of probability measures in terms of continuity sets, we obtain by Theorem 6 that

$$
\lim_{T \to \infty} P_{T,w}(\hat{G}_\varepsilon) = P_{\zeta}(\hat{G}_\varepsilon)
$$

(6.4)

for all but at most countably many \( \varepsilon > 0 \). Moreover, inequalities (6.1) imply the inclusion \( G_\varepsilon \subset \hat{G}_\varepsilon \). Thus, by (6.2), the inequality \( P_{\zeta}(\hat{G}_\varepsilon) > 0 \) holds. This, the definitions of \( P_{T,w} \) and \( \hat{G}_\varepsilon \), and (6.4) prove the second assertion of the theorem. \( \square \)

**Acknowledgements**

The research of the first author is funded by the European Social Fund (project No. 09.3.3-LMT-K-712-01-0037) under grant agreement with the Research Council of Lithuania (LMT LT).
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