Universality of Zeta-Functions of Cusp Forms and Non-Trivial Zeros of the Riemann Zeta-Function

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Abstract. It is known that zeta-functions $\zeta(s,F)$ of normalized Hecke-eigen cusp forms $F$ are universal in the Voronin sense, i.e., their shifts $\zeta(s+i\tau,F)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. In the paper, under a weak form of the Montgomery pair correlation conjecture, it is proved that the shifts $\zeta(s+i\gamma_kh,F)$, where $\gamma_1 < \gamma_2 < \ldots$ is a sequence of imaginary parts of non-trivial zeros of the Riemann zeta function and $h > 0$, also approximate a wide class of analytic functions.

Keywords: Montgomery pair correlation conjecture, Riemann zeta-function, zeta-function of cusp form, universality.

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1 Introduction

We start with some definitions. Let
\[ SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \]
be the full modular group. Suppose that \( F(z) \) is a holomorphic function in the upper half-plane, and, for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \), satisfies the functional equation
\[ F\left( \frac{az + b}{cz + d} \right) = (cz + d)^\kappa F(z) \]
for some \( \kappa \in 2\mathbb{N} \). Then \( F(z) \) has the Fourier series expansion at infinity
\[ F(z) = \sum_{m=-\infty}^{\infty} c(m) e^{2\pi i mz}. \]
If \( c(m) = 0 \) for \( m < 0 \), then \( F(z) \) is called a modular form of weight \( \kappa \). If the modular form \( F(s) \) has the Fourier series expansion at infinity
\[ F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i mz}, \]
then it is called a cusp form of weight \( \kappa \) for the full modular group.

Suppose that \( F(z) \) is a cusp form of weight \( \kappa \) for the full modular group. Then the zeta-function
\[ \zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}, \quad s = \sigma + it \]
can be attached to \( F(z) \). The latter series, in view of the estimate
\[ c(m) \ll m^{\frac{\kappa-1}{2}}, \]
is absolutely convergent for \( \sigma > \frac{\kappa+1}{2} \). Moreover, it has analytic continuation to an entire function.

We additionally require that the function \( F(z) \) would be the Hecke-eigen cusp form, i.e., that \( F(z) \) would be the eigenfunction of all Hecke operators \( T_m \),
\[ T_m f(z) = m^{\kappa-1} \sum_{\substack{a, d > 0 \\ ad = m}} \frac{1}{d^\kappa} \sum_{b \pmod{d}} F\left( \frac{az + b}{d} \right), \quad m \in \mathbb{N}. \]
Then the form \( F(z) \) can be normalized, thus, we may suppose that \( c(1) = 1 \).

In the sequel, we suppose that \( F(z) \) is a normalized Hecke-eigen cusp form of weight \( \kappa \). In this case, the zeta-function \( \zeta(s, F) \) has, for \( \sigma > \frac{\kappa+1}{2} \), the Euler product representation over primes
\[ \zeta(s, F) = \prod_p (1 - \alpha(p)/p^s)^{-1} (1 - \beta(p)/p^s)^{-1}, \]
where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers such that $\alpha(p) + \beta(p) = c(p)$.

In [8], it was proved that the function $\zeta(s, F)$ is universal in the Voronin sense, i.e., a wide class of analytic functions is approximated by shifts $\zeta(s + i\tau, F)$, $\tau \in \mathbb{R}$. More precisely, let $D_\kappa = \{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$. Denote by $K_F$ the class of compact subsets of the strip $D_\kappa$ with connected complements, and by $H_0(F)(K)$ with $K \in K_F$ the class of continuous non-vanishing functions on $K$ that are analytic in the interior of $K$. Then the main result of [8] is the following statement.

**Theorem 1.** Suppose that $K \in K_F$ and $f(s) \in H_0(F)(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon\} > 0.$$ 

Here $\text{meas}A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

In the latter theorem, $\tau$ in shifts $\zeta(s + i\tau, F)$ takes arbitrary real values, therefore, the theorem is of continuous type. Also, Theorem 1 has a discrete version when $\tau$ in $\zeta(s + i\tau, F)$ takes values from certain discrete sets. The classical discrete set is an arithmetical progression $\{kh : k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$, where $h > 0$ is a fixed number. Discrete universality theorems for the function $\zeta(s, F)$ were considered in [9] and [10], and the following statement has been obtained.

**Theorem 2.** Suppose that $K \in K_F$, $f(s) \in H_0(F)(K)$ and $h > 0$. Then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\{0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ikh, F) - f(s)| < \varepsilon\} > 0.$$ 

Here $\#A$ denotes the cardinality of a set $A$, and $N$ runs over non-negative integers.

In [12], more general shifts $\zeta(s + i\varphi(k), F)$ were used. Here $\varphi(t)$ is a real-valued positive increasing function on $[k_0 - \frac{1}{2}, \infty)$, $k_0 \in \mathbb{N}$, having a continuous derivative $\varphi'(t)$ satisfying the estimate

$$\varphi(2t) \max_{t \leq u \leq 2t} \left( \frac{1}{\varphi'(u)} + \varphi'(u) \right) \ll t,$$

and such that the sequence $\{a\varphi(k) : k \geq k_0\}$ with every $a \in \mathbb{R} \setminus \{0\}$ is uniformly distributed modulo 1.

In [13], a joint version of a theorem from [12] has been proved.

The aim of this paper is an extension of Theorem 2 for the discrete set related to non-trivial zeros of the Riemann zeta-function $\zeta(s)$ which is defined, for $\sigma > 1$, by the series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$
and has a meromorphic continuation to the whole complex plane. The function
\( \zeta(s) \) has infinitely many so-called non-trivial zeros \( \gamma = \beta + i\gamma \) lying in the strip \( \{ s \in \mathbb{C} : 0 < \sigma < 1 \} \). By the Riemann hypothesis, all non-trivial zeros of \( \zeta(s) \) lie on the critical line \( \sigma = \frac{1}{2} \).

Thus, let \( 0 < \gamma_1 < \gamma_2 < \ldots \leq \gamma_k \leq \ldots \) be the sequence of imaginary parts of non-trivial zeros of the function \( \zeta(s) \). We will use a hypothesis on the distribution of the sequence \( \{ \gamma_k : k \in \mathbb{N} \} \), namely, we suppose that, for \( c > 0 \),
\[
\sum_{\gamma_k \leq T} \sum_{\gamma_l \leq T} \frac{1}{|\gamma_k - \gamma_l|} \lesssim T \log T, \quad T \to \infty. \tag{1.1}
\]
The latter estimate is implied by the famous Montgomery pair correlation conjecture [16]. The main result of the paper is the following theorem.

**Theorem 3.** Suppose that the estimate (1.1) is true. Let \( K \in \mathcal{K}_F, f(s) \in H_{0F}(K) \) and \( h > 0 \). Then, for every \( \varepsilon > 0 \),
\[
\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - f(s)| < \varepsilon \right\} > 0.
\]
Moreover, the limit
\[
\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - f(s)| < \varepsilon \right\} > 0
\]
exists for all but at most countably many \( \varepsilon > 0 \).

Theorem 3 with the Riemann hypothesis in place of (1.1) was proved in [4] by using [3].

We recall that the condition (1.1) for the first-time was applied in [5] for the approximation by shifts \( \zeta(s + i\gamma_k h) \), and in [7] for joint approximation by shifts \( \langle \zeta(s + i\gamma_k h), \zeta(s + i\gamma_k h, \alpha) \rangle \), where \( \zeta(s, \alpha) \) is the Hurwitz zeta-function
\[
\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1
\]
with transcendental parameter \( \alpha \). In [11], the joint approximation by shifts of Dirichlet \( L \)-functions involving the sequence \( \{ \gamma_k \} \) was discussed. Finally, the paper [1] is devoted to a generalization of [7] for shifts of the periodic and periodic Hurwitz zeta-functions.

For the proof of Theorem 3, we will apply some results from [5] and [8]. On the mentioned results, we will construct a probabilistic model.

## 2 Probabilistic model

Denote by \( B(\mathbb{X}) \) the Borel \( \sigma \)-field of the space \( \mathbb{X} \), and let \( H(D_F) \) be the space of analytic functions on \( D_F \) endowed with the topology of uniform convergence.
on compacta. In this section, we will consider the weak convergence as \( N \to \infty \) for
\[
P_{N,F}(A) \overset{\text{def}}{=} \frac{1}{N} \# \{ 1 \leq k \leq N : \zeta(s + i\gamma_k h, F) \in A \}, \quad A \in \mathcal{B}(H(D_F)).
\]

To state a limit theorem for \( P_{N,F} \), we need some notation. Denote by \( \gamma \) the unit circle on the complex plane, by \( \mathbb{P} \) the set of all prime numbers, and define the set \( \Omega = \prod_{p \in \mathbb{P}} \gamma_p \), where \( \gamma_p = \gamma \) for all \( p \in \mathbb{P} \). With the product topology and operation of pointwise multiplication, the infinite-dimensional torus \( \Omega \) is a compact topological Abelian group. Therefore, on \( (\Omega, \mathcal{B}(\Omega)) \), the probability Haar measure \( m_H \) exists, and we have the probability space \( (\Omega, \mathcal{B}(\Omega), m_H) \).

Denote by \( \omega = (\omega(p) : p \in \mathbb{P}) \) the elements of the torus \( \Omega \), and on the above probability space define the \( H(D_F) \)-valued random element
\[
\zeta(s, \omega, F) = \prod_{p \in \mathbb{P}} \left( 1 - \frac{\alpha(p)\omega(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta(p)\omega(p)}{p^s} \right)^{-1}.
\]

We note that the latter infinite product is uniformly convergent on compact subsets of the strip \( D_F \) for almost all \( \omega \in \Omega \), thus, it defines an \( H(D_F) \)-valued random element. Denote by \( P_{\zeta,F} \) the distribution of the random element \( \zeta(s, \omega, F) \), i.e., for \( A \in \mathcal{B}(H(D_F)) \),
\[
P_{\zeta,F}(A) = m_H \{ \omega \in \Omega : \zeta(s, \omega, F) \in A \}.
\]

We will prove the following statement

**Theorem 4.** Suppose that the estimate (1.1) is true. Then \( P_{N,F} \) converges weakly to the measure \( P_{\zeta,F} \) as \( N \to \infty \).

The proof of Theorem 4 consists from three limit theorems that will be stated as separate lemmas.

For \( A \in \mathcal{B}(\Omega) \), define
\[
Q_N(A) = \frac{1}{N} \# \{ 1 \leq k \leq N : (p^{-i\gamma_k h} : p \in \mathbb{P}) \in A \}.
\]

**Lemma 1.** \( Q_N \) converges weakly to the Haar measure \( m_H \) as \( N \to \infty \).

The lemma is proved in [5] by using the Fourier transform method. For this, the uniform distribution modulo 1 of the sequence \( \{a\gamma_k : k \in \mathbb{N}\} \) with every \( a \in \mathbb{R} \setminus \{0\} \) is applied.

The next lemma deals with absolutely convergent Dirichlet series. Let \( \theta > \frac{1}{2} \) be a fixed number, for \( m, n \in \mathbb{N} \), let
\[
v_n(m) = \exp \left\{ -\left( \frac{m}{n} \right)^\theta \right\}, \quad \zeta_n(s, F) = \sum_{m=1}^\infty \frac{c(m)v_n(m)}{m^s}.
\]

Then it is known [8] that the latter series is absolutely convergent for \( \sigma > \frac{\kappa}{2} \). Consider the mapping \( u_{n,F} : \Omega \to H(D_F) \) given by \( u_{n,F}(\omega) = \zeta_n(s, \omega, F) \), where
\[
\zeta_n(s, \omega, F) = \sum_{m=1}^\infty \frac{c(m)\omega(m)v_n(m)}{m^s}, \quad \omega(m) = \prod_{p^i|m, p^{i+1}|m} \omega^l(p), \quad m \in \mathbb{N}.
\]
Obviously, the series for $\zeta_n(s, \omega, F)$ is also absolutely convergent for $\sigma > \frac{\kappa}{2}$. Therefore, the mapping $u_{n,F}$ is continuous, hence it is $(\mathcal{B}(\Omega), \mathcal{B}(H(D_F)))$-measurable. Therefore, the Haar measure $m_H$ defines the unique probability measure $V_{n,F} = m_H u_{n,F}^{-1}$ on $(H(D_F), \mathcal{B}(H(D_F)))$, where, for $A \in \mathcal{B}(H(D_F))$,

$$V_{n,F}(A) = m_H u_{n,F}^{-1}(A) = m_H(u_{n,F}^{-1}A).$$

For $A \in \mathcal{B}(H(D_F))$, set

$$P_{N,n,F}(A) = \frac{1}{N} \# \left\{ 1 \leq k \leq N : \zeta_n(s + i\gamma_k h, F) \in A \right\}.$$

**Lemma 2.** $P_{N,n,F}$ converges weakly to the measure $V_{n,F}$ as $N \to \infty$.

**Proof.** By the definitions of $Q_N$ and $P_{N,n,F}$, we have

$$P_{N,n,F}(A) = \frac{1}{N} \# \left\{ 1 \leq k \leq N : (p^{-i\gamma_k h} : p \in \mathbb{P}) \in u_{n,F}^{-1}A \right\} = Q_N(u_{n,F}^{-1}A).$$

Thus, $P_{N,n,F} = Q_N u_{n,F}^{-1}$. Therefore, the lemma is a corollary of Lemma 1, continuity of $u_{n,F}$ and Theorem 5.1 of [2].

The weak convergence of the measure $V_{n,F}$ as $n \to \infty$ is very important for the proof of Theorem 4. The following assertion is true.

**Lemma 3.** $V_{n,F}$ converges weakly to the measure $P_{\zeta,F}$ as $n \to \infty$. Moreover, the support of $P_{\zeta,F}$ is the set

$$S_F = \left\{ g \in H(D_F) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \right\}.$$

**Proof.** The lemma is a result of [6] and [8] because $V_{n,F}$, as $n \to \infty$, and

$$\frac{1}{T} \text{meas}\{ \tau \in [0, T] : \zeta(s + i\tau, F) \in A \}, \ A \in \mathcal{B}(H(D_F)),$$

as $T \to \infty$, have the same limit measure $P_{\zeta,F}$. □

To prove Theorem 4, it remains to show that the limit measure of $P_{N,F}$ as $N \to \infty$ coincides with that of $V_{n,F}$ as $n \to \infty$. For this, some mean square estimates will be applied. For convenience, we recall the Gallagher lemma which connects discrete and continuous mean squares of certain functions.

**Lemma 4.** Let $T_0$ and $T \geq \delta > 0$ be real numbers, and $\mathcal{T} \neq \emptyset$ be a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Define

$$N_\delta(x) = \sum_{t \in \mathcal{T}, |t-x| < \delta} 1$$

and let $S(t)$ be a complex-valued continuous function on $[T_0, T_0 + T]$ having a continuous derivative on $(T_0, T_0 + T)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leq \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(t)|^2dt + \left( \int_{T_0}^{T_0+T} |S(t)|^2dt \int_{T_0}^{T_0+T} |S'(t)|^2dt \right)^{\frac{1}{2}}.$$
The proof of the lemma can be found in [15, Lemma 1.4]. 

Now, we recall a metric in the space $H(D_F)$. For $g_1, g_2 \in H(D_F)$, let

$$
\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \sup_{s \in K_l} \frac{|g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},
$$

where $\{K_l : l \in \mathbb{N}\}$ is a sequence of compact subsets of the strip $D_F$ such that $D_F = \bigcup_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K$ is a compact subset of $D_F$, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then $\rho$ is a metric on $H(D_F)$ that induces the topology of uniform convergence on compacta.

**Lemma 5.** Suppose that the estimate (1.1) is true. Then the equality

$$
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \rho(\zeta(s + i\gamma_k h, F), \zeta_n(s + i\gamma_k h, F)) = 0
$$

holds.

**Proof.** We start with some remarks on the mean squares of the function $\zeta(s, F)$. It is well known that, for fixed $\sigma$, $\frac{5}{2} < \sigma < \frac{\sigma+1}{2}$, the bound

$$
\int_{0}^{T} |\zeta(\sigma + it, F)|^2 dt \ll_{\sigma} T
$$

is true. Hence, it follows for the same $\sigma$ that, for $\tau \in \mathbb{R}$,

$$
\int_{0}^{T} |\zeta(\sigma + i\tau + it, F)|^2 dt \ll_{\sigma} T(1 + |\tau|). \quad (2.1)
$$

Moreover, the Cauchy integral formula together with (2.1) leads to

$$
\int_{0}^{T} |\zeta'(\sigma + i\tau + it, F)|^2 dt \ll_{\sigma} T(1 + |\tau|). \quad (2.2)
$$

Now, we apply Lemma 4. It is known that $\gamma_k \sim \frac{2\pi k}{\log k}$ as $k \to \infty$. Therefore, $\gamma_k \leq \frac{ck}{\log k}$ with some $c > 0$ for all $k \geq 2$. In Lemma 4, we take $T = \{\gamma_1 h, \ldots, \gamma_N h\}$, $\delta = h \left(\log \frac{N}{c\log N}\right)^{-1}$, $T_0 = \gamma_1 h - \frac{\delta}{2}$ and $T = \gamma_N h - T_0 + \frac{\delta}{2}$. Then, in view of (1.1), we find that

$$
\sum_{k=1}^{N} N_\delta(\gamma_k h) = \sum_{k=1}^{N} \sum_{\gamma_1 \leq \gamma_k \leq \frac{\gamma_1 h}{N}} \sum_{0 < \gamma_1, \gamma_k \leq \frac{\gamma_1 h}{N}} 1 \ll_{\log N} N_\delta(\gamma_k h)
$$

Thus, applying Lemma 4 for the function $\zeta(\sigma + i\tau + i\gamma_k h, F)$, and, taking into account the estimates (2.1) and (2.2), we obtain

$$
\sum_{k=1}^{N} |\zeta(\sigma + i\tau + i\gamma_k h, F)| = \sum_{k=1}^{N} \left(N_\delta(\gamma_k h)N_\delta^{-1}(\gamma_k h)\right)^{\frac{1}{2}} |\zeta(\sigma + i\tau + i\gamma_k h, F)|
$$
We take an arbitrary fixed compact subset $K$ of the strip $D_F$, denote the points of $K$ by $s = \sigma + iv$, fix $\varepsilon > 0$ such $\frac{\kappa}{2} + 2\varepsilon \leq \sigma \leq \frac{\kappa+1}{2} - \varepsilon$ for $s \in K$, and choose $\theta_1 = \sigma - \varepsilon - \frac{\kappa}{2}$ and $\theta = \frac{\kappa}{2} + \varepsilon$. Then the representation (2.4) shows that, for $s \in K$,

$$
\zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F) \ll \int_{-\infty}^{\infty} \left| \zeta(s + i\gamma_k h - \theta_1 + i\tau, F) \right| \left| \frac{l_n(-\theta_1 + i\tau)}{|-\theta_1 + i\tau|} \right| d\tau.
$$

Hence, after a shift $\tau + \nu \to \tau$, we have

$$
\zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F) \ll \int_{-\infty}^{\infty} \left| \zeta \left( \frac{\kappa}{2} + \varepsilon + i(\tau + \gamma_k h), F \right) \right| \times \left| \frac{l_n(\frac{\kappa}{2} + \varepsilon - s + i\tau)}{|\frac{\kappa}{2} + \varepsilon - s + i\tau|} \right| d\tau.
$$
Therefore,

$$\frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F)| \ll \int_{-\infty}^{\infty} \left( \frac{1}{N} \sum_{k=1}^{N} \left| \zeta \left( \frac{k}{2} + \epsilon + i(\tau + \gamma_k h), F \right) \right| \sup_{s \in K} \left| l_n \left( \frac{k}{2} + \epsilon - s + i\tau \right) \right| \right) d\tau. \quad (2.5)$$

It is well known that, uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$ with arbitrary $\sigma_1 < \sigma_2$,

$$\Gamma(\sigma + i\tau) \ll \exp\{-c|\tau|\}, \quad c > 0.$$  

Thus, taking into account the definition of the function $l_n(s)$, we find that, for $s \in K$,

$$\frac{l_n \left( \frac{k}{2} + \epsilon - s + i\tau \right)}{\frac{k}{2} + \epsilon - s + i\tau} \ll n^{-\epsilon} \exp\left\{ - \frac{c|\tau - v|}{\theta} \right\} \ll_K n^{-\epsilon} \exp\{-c|\tau|\}.$$  

Therefore, by (2.5) and (2.3),

$$\frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} \left| \zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F) \right| \ll_K n^{-\epsilon} \int_{-\infty}^{\infty} (1 + |\tau|) \exp\{-c|\tau|\} d\tau \ll_K n^{-\epsilon}.$$  

This shows that, for every compact set $K \subset D_F$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} \left| \zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F) \right| = 0,$$

and the assertion of the lemma follows from the definition of the metric $\rho$.  \( \Box \)

Now, we are in position to prove Theorem 4.

**Proof of Theorem 4.** Let $\xi_N$ be a random variable on a certain probability space $(\hat{\Omega}, \mathcal{A}, \mu)$ with the distribution

$$\mu\{\xi_N = \gamma_k h\} = \frac{1}{N}, \quad k = 1, \ldots, N.$$  

Denote by $X_{n,F}$ the $H(D_F)$-valued random element with the distribution $V_{n,F}$, where $V_{n,F}$ is the limit measure in Lemma 2, and, on the probability space $(\hat{\Omega}, \mathcal{A}, \mu)$, define the $H(D_F)$-valued random element

$$X_{N,n,F} = X_{N,n,F}(s) = \zeta_n(s + i\xi_N, F).$$  

Then, in view of Lemma 2,

$$X_{N,n,F} \xrightarrow{D} X_{n,F}.$$  

(2.6)
By Lemma 2, the measure $V_{n,F}$ is weakly convergent to $P_{\zeta,F}$ as $n \to \infty$. Thus,

$$X_{n,F} \xrightarrow{D} P_{\zeta,F}. \quad (2.7)$$

On the above probability space, define one more $H(D_F)$-valued random element

$$Y_{N,F} = Y_{N,F}(s) = \zeta(s + i\xi_N,F).$$

Then, applying Lemma 5, we find that, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \mu\{\rho(Y_{N,F}, X_{N,n,F}) \geq \varepsilon\} \leq \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N\varepsilon} \sum_{k=1}^{N} \rho(\zeta(s + i\gamma_k h, F), \zeta_n(s + i\gamma_k h, F)) = 0.$$ 

This equality together with (2.6) and (2.7) shows that all hypotheses of Theorem 4.2 in [2] are satisfied. Therefore, we have

$$Y_{N,F} \xrightarrow{D} P_{\zeta,F},$$

in other words, $P_{N,F}$ converges weakly to $P_{\zeta,F}$ as $N \to \infty$. The theorem is proved.

3 Proof of Theorem 3

The proof of Theorem 3 is quite standard, and is based on Theorem 4 and the Mergelyan theorem on the approximation of analytic functions by polynomials [14].

Proof of Theorem 3. By the mentioned Mergelyan theorem, there exists a polynomial $p_\varepsilon(s)$ such that

$$\sup_{s \in K} \left| f(s) - e^{p_\varepsilon(s)} \right| < \frac{\varepsilon}{2}. \quad (3.1)$$

Define the set

$$G_\varepsilon = \left\{ g \in H(D_F) : \sup_{s \in K} |g(s) - e^{p_\varepsilon(s)}| < \frac{\varepsilon}{2} \right\}.$$ 

Clearly, $e^{p_\varepsilon(s)} \in S$. Therefore, in virtue of Lemma 3, the set $G_\varepsilon$ is an open neighbourhood of an element of the support of the measure $P_{\zeta,F}$. Hence, by a property of the support,

$$P_{\zeta,F}(G_\varepsilon) > 0, \quad (3.2)$$

and Theorem 4 together with the equivalent of weak convergence of probability measures in terms of open sets [2, Theorem 2.1] implies

$$\liminf_{N \to \infty} P_{N,F}(G_\varepsilon) \geq P_{\zeta,F}(G_\varepsilon) > 0.$$
This, the definitions of $P_{N,F}$ and $G_\varepsilon$, and (3.1) prove the first assertion of the theorem.

For the proof of the second assertion of the theorem, define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D_F) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$  

Then the boundary $\partial \hat{G}_\varepsilon$ lies in the set

$$\left\{ g \in H(D_F) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\},$$

therefore, $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \emptyset$ for different positive $\varepsilon_1$ and $\varepsilon_2$. This remark implies that the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,F}$, i.e., $P_{\zeta,F}(\partial \hat{G}_\varepsilon) = 0$, for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 4 together with the equivalent of weak convergence of probability measures in terms of continuity sets [2, Theorem 2.1] gives the equality

$$\lim_{N \to \infty} P_{N,F}(\hat{G}_\varepsilon) = P_{\zeta,F}(\hat{G}_\varepsilon) \quad (3.3)$$

for all but at most countably many $\varepsilon > 0$. The definitions of the sets $G_\varepsilon$ and $\hat{G}_\varepsilon$, and inequality (3.1) imply the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$. Hence, in view of (3.2), we have $P_{\zeta,F}(\hat{G}_\varepsilon) > 0$. The latter inequality, the definitions of $P_{N,F}$ and $\hat{G}_\varepsilon$, and (3.3) prove the second assertion of the theorem. The theorem is proved.

References


