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Multi-Step Algorithms for Solving EPs

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Abstract. The paper introduces and analysizes the convergence of two multi-step proximal-like algorithms for pseudomonotone and Lipschitz-type continuous equilibrium problems in a real Hilbert space. The algorithms are combinations between the multi-step proximal-like method and Mann or Halpern iterations. The weakly and strongly convergent theorems are established with the prior knowledge of two Lipschitz-type continuous constants. Moreover, by choosing two sequences of suitable stepsizes, we also show that the multi-step proximal-like algorithm for strongly pseudomonotone and Lipschitz-type continuous equilibrium problems where the construction of solution approximations and the establishing of its convergence do not require the prior knowledge of strongly pseudomonotone and Lipschitz-type continuous constants of bifunctions. Finally, several numerical examples are reported to illustrate the convergence and the performance of the proposed algorithms over classical extragradient-like algorithms.

Keywords: proximal-like method, extragradient method, equilibrium problem, multi-step method, Lipschitz-type continuous.

AMS Subject Classification: 65J15; 47H05; 47J25; 47J20; 91B50.

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space and C be a nonempty closed convex subset of $\mathcal{H}$. Let f be a bifunction from $C \times C$ to the set of real numbers $\mathcal{R}$ such that $f(x,x) = 0$ for all $x \in C$. The equilibrium problem, shortly EP, for the bifunction f on C is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C. \tag{1.1}$$

Let us denote by $EP(f, C)$ the solution set of EP (1.1). Mathematically, EP (1.1) is a generalization of many previously known mathematical models as variational inequalities, optimization problems, fixed point problems and Nash equilibrium problems [19]. In recent years, EP (1.1) has recieved a lot of attention by many authors because it allows us to unify all these particular problems in a convenient way. Several methods have been proposed for solving EP (1.1), for instance, the ergodic iteration method [3], the dual extragradient method [21], the proximal point method [18], the auxiliary problem principle method [16] and the gap function method [17], the subgradient method [23], the proximal-like method (also, called the extragradient method) [7, 24], and other hybrid methods [1, 2, 4, 8, 9, 10, 11, 12, 25].

A special case of EP (1.1) is variational inequality problem, shortly VIP, when the bifunction f defined by $f(x, y) = \langle Ax, y - x\rangle$, where $A : C \to \mathcal{H}$ is an operator, i.e., find $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \ \forall y \in C. \tag{1.2}$$

The solution set of VIP (1.2) is denoted by $VI(A, C)$. The projection method plays an important role in solving VIP. The simplest projection method is the gradient method, however, the convergence of this method requires a slightly strong assumption that operators are strongly monotone or inverse strongly monotone. To overcome this assumption, Korpelevich [14] introduced the extragradient method for solving saddle point problems, and then, it was extended to VIPs for the class of monotone (even, pseudomonotone) and Lipschitz continuous operators. Due to this reason, the extragradient method has recieved a lot of attention by many authors who have modified it via many ways, for instance [5, 15] and the references therein.

Recently, Zykina and Melenchuk [27] have introduced a two-step extragradient method with a suitable fixed stepsize $\beta > 0$ for VIPs in Euclidean spaces. In this method, three projections on feasible set needed to be computed per each iteration. More precisely, the algorithm in [27] is described as follows:

$$\begin{cases} \bar{x}_n = P_C(x_n - \beta A(x_n)), \\ \tilde{x}_n = P_C(\bar{x}_n - \beta A(\bar{x}_n)), \\ x_{n+1} = P_C(x_n - \beta A(\tilde{x}_n)). \end{cases} \tag{1.3}$$

Inspired of the result of Zykina and Melenchuk, the authors in [20] first introduced a more general version of process (1.3) for solving VIP (1.2) in the real Euclidean space where two sequences of suitable variant stepsizes $\{\alpha_n\}$ and $\{\beta_n\}$ have been used. Precisely, this version is designed as follows:

$$\begin{cases} \bar{x}_n = P_C(x_n - \alpha_n A(x_n)), \\ \tilde{x}_n = P_C(\bar{x}_n - \beta_n A(\bar{x}_n)), \\ x_{n+1} = P_C(x_n - \beta_n A(\tilde{x}_n)). \end{cases} \tag{1.4}$$

It was implied in [20] that process (1.4) contains both the classical extragradient method [14] and the two-step extragradient method (1.3). The authors in [20] considered a general version of process (1.4) for EPs and proved its convergence.

The computational performance of all the two-step algorithms over the classical extragradient method have been illustrated by several numerical experiments for solving resource management problems in [26, 28] and others in [20].

The motivation and inspriation by the advantage of the mentioned algorithms, in this paper, we develop continuously the algorithms in [20, 27] for solving EPs in $\mathcal{H}$. We first introduce a weakly convergent algorithm, called the Mann multi-step proximal-like algorithm, which combines the mul-step proximal-like method with Mann-like iteration. After that, in order to the strong convergence, we repalce the Mann-like iteration by the Halpern iteration and obtain the Halpern multi-step proximal-like algorithm. Comparing with the current algorithms the fundamental difference here is that, two convergent theorems are established only under the hypotheses of pseudomonotonicity and Lipschitz-type continuity of bifunctions. The construction of iterative sequences in the first two algorithms and the proving of their convergence require that two Lipschitz-type continuous constants need to be known. Next, we use two sequences of suitable stepsizes and proposed an iteration algorithm for strongly pseudomonotone and Lipschitz-type continuous bifunctions. In this algorithm, the prior knowledge of strong pseudomonotonicity and Lipschitz-type continuous constants is not requried. Finally, we report several numerical experiments to illustrate the performance of the proposed algorithms over the classical extragradient - like methods.

This paper is organized as follows: In Section 2, we recall some definitions and preliminary results used in the paper. Sections 3 and 4 present respectively two Mann and Halpern multi-step proximal-like algorithms, and analysize their convergence. In Section 5, we propose and prove the convergence of the multi-step proximal-like algorithm for solving strongly pseudomonotone and Lipschitz-type continuous equilibrium problems. Finally, Section 6 reports numerical experiments to illustrate the convergence and compare them with several other algorithms.

2 Preliminaries

Let C be a nonempty closed convex subset of $\mathcal{H}$. We begin with some concepts of monotonicity of a bifunction, see [19] for more details.
A bifunction $f : C \times C \to \mathcal{R}$ is said to be:

(i) *strongly monotone* on C, if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma||x - y||^2, \ \forall x, y \in C;$$

(ii) *monotone* on C, if

$$f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C;$$

(iii) *pseudomonotone* on C, if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0, \ \forall x, y \in C;$$

(iv) *strongly pseudomonotone* on C, if there exists a constant $\gamma > 0$ such that

$$f(x,y) \geq 0 \Longrightarrow f(y,x) \leq -\gamma\|x-y\|^2,\ \forall x,y \in C;$$

(v) *Lipschitz-type continuous* on C, if there exist two positive constants c_1, c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2,\ \forall x,y,z \in C.$$

From the definitions above, we see that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii). If $A : C \to H$ is L - Lipschitz continuous, the bifunction $f(x,y) = \langle Ax, y-x\rangle$ satisfies the Lipschitz-type continuous condition with $c_1 = c_2 = \frac{L}{2}$.

Dealing with the convergence of the first proposed algorithms in this paper, we consider the following conditions imposed on the bifunction $f : C \times C \to \mathcal{R}$:
(A1) f is pseudomonotone on C and $f(x,x) = 0$ for all $x \in C$;
(A2) f is Lipschitz-type continuous on C with the constants c_1, c_2;
(A3) $f(x,.)$ is convex and subdifferentiable on C for every fixed $x \in C$;
(A4) The solution set $EP(f,C)$ is nonempty;
(A5) $f(.,y)$ is weakly sequencially upper semicontinuous on C with every fixed $y \in C$, i.e., $\limsup_{n\to\infty} f(x_n,y) \leq f(x,y)$ for each sequence $\{x_n\} \subset C$ converging weakly to x.

It is easy to show that under assumptions (A1) and (A3), the solution set $EP(f,C)$ is closed and convex.

Recall that the metric projection $P_C : \mathcal{H} \to C$ is defined by

$$P_C x = \arg\min\{\|y-x\| : y \in C\}.$$

Since C is nonempty, closed and convex, $P_C x$ exists and is unique. It is also known that P_C has the following characteristic properties.

Lemma 1. *Let $P_C : \mathcal{H} \to C$ be the metric projection from $\mathcal{H}$ onto C. Then*

i. *For all $x \in C, y \in \mathcal{H}$,*

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x-y\|^2.$$

ii. *$z = P_C x$ if and only if*

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C.$$

Let $g : C \to \mathcal{R}$ be a function. The subdifferential of g at x is defined by

$$\partial g(x) = \{w \in \mathcal{H} : g(y) - g(x) \geq \langle w, y-x\rangle,\ \forall y \in C\}.$$

A function $\varphi : \mathcal{H} \to \mathcal{R}$ is called *weakly lower semicontinuous* at $x \in \mathcal{H}$ if for any sequence $\{x_n\}$ in $\mathcal{H}$ converges weakly to x then $\varphi(x) \leq \liminf_{n\to\infty}\varphi(x_n)$. It is well-known that the functional $\varphi(x) := ||x||^2$ is convex and weakly lower semicontinuous.

Recall the proximal mapping of a proper, convex and lower semicontinuous function $g : C \to \mathcal{R}$ with a parameter $\lambda > 0$ as follows:

$$prox_{\lambda g}(x) = \arg\min \left\{ \lambda g(y) + \frac{1}{2}||x - y||^2 : y \in C \right\}, \ x \in \mathcal{H}.$$

The following is a property of the proximal mapping, see [6] for more details.

Lemma 2. *[6, Proposition 12.26] For all $\forall x \in \mathcal{H}, \ y \in C$ and $\lambda > 0$,*

$$\lambda \{g(y) - g(prox_{\lambda g}(x))\} \geq \langle x - prox_{\lambda g}(x), y - prox_{\lambda g}(x) \rangle .$$

We need the following technical lemmas.

Lemma 3. *[22, Theorem 15.5] Let $\{\tau_n\}$ be a sequence which converges to 0 as $n \to \infty$ and $0 \leq \tau_n < 1$ for all $n \geq 1$. Then*

$$\sum_{n=1}^{\infty} \tau_n = +\infty \Longleftrightarrow \prod_{n=1}^{\infty}(1 - \tau_n) = \lim_{n\to\infty} \prod_{k=1}^{n}(1 - \tau_k) = 0.$$

Lemma 4. *[13, Remark 4.4] Let $\{a_n\}$ be a sequence of non-negative real numbers. Suppose that for any integer m, there exists an integer p such that $p \geq m$ and $a_p \leq a_{p+1}$. Let n_0 be an integer such that $a_{n_0} \leq a_{n_0+1}$ and define, for all integer $n \geq n_0$,*

$$\tau(n) = \max \{k \in N : n_0 \leq k \leq n, \ a_k \leq a_{k+1}\} .$$

Then $0 \leq a_n \leq a_{\tau(n)+1}$ for all $n \geq n_0$. Furthermore, the sequence $\{\tau(n)\}_{n \geq n_0}$ is non-decreasing and tends to $+\infty$ as $n \to \infty$.

3 Mann multi-step proximal-like algorithm

In this section, we introduce a weakly convergent algorithm which combines the mul-step proximal-like method with the Mann type iterative method for solving EPs, so-called the Mann Multi-step Proximal-like Algorithm (Mann MPA). The Mann MPA is designed as follows:

Algorithm 1 [Mann MPA].
Initialization. Choose $x_0 \in C$ and control parameters $\lambda_n > 0, \ \rho_n > 0, \ \alpha_n \in (0, 1)$.
Step 1. Compute $y_n = prox_{\lambda_n f(x_n,.)}(x_n), \ z_n = prox_{\rho_n f(y_n,.)}(y_n), \ t_n = prox_{\rho_n f(z_n,.)}(x_n)$.
Step 2. Compute $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)t_n$. Set $n := n + 1$ and go back **Step 1**.

When $\alpha_n = 0$ and $z_n = y_n$, we find again the extragradient method for solving equilibrium problems proposed by Quoc et al. in [24]. In order to obtain the estimates in Lemmas 5 and 6 below, we choose two parameter sequences $\{\rho_n\}$ and $\{\lambda_n\}$ satisfing the following conditions:

(B1) $0 < \rho_n \leq \overline{\rho} < \min \{1/6c_1, 1/4c_2, 1/2c_1 + 3c_2\}$.

(B2) $0 \leq \lambda_n \leq \rho_n$.

Throughout this paper, we denote Δ_n and T_n by

$$\Delta_n = \min\{1 - 3\rho_n c_2 - 2\rho_n c_1, 1 - 4\rho_n c_2\},$$

$$T_n = (1-6\rho_n c_1)\|y_n - x_n\|^2 + (1 - 2\rho_n c_1 - 3\rho_n c_2)\|y_n - z_n\|^2 + \frac{\Delta_n}{2}\|t_n - z_n\|^2.$$

Under Condition (B1), there exists a number Δ^* such that $0 < \Delta^* \leq \Delta_n$ for all $n \geq 0$. We have the following lemma which plays an important role in establishing the convergence of all the algorithms in this paper.

Lemma 5. *Under Conditions (A1)–(A4) and (B1)–(B2), there holds the following estimate for all $x^* \in EP(f, C)$ and $n \geq 0$,*

$$||t_n - x^*||^2 - 2\rho_n f(z_n, x^*) \leq ||x_n - x^*||^2 - T_n.$$

Proof. It follows from $t_n = prox_{\rho_n f(z_n,.)}(x_n)$ and Lemma 2 with $y = x^*$ that

$$\rho_n \{f(z_n, x^*) - f(z_n, t_n)\} \geq \langle x_n - t_n, x^* - t_n \rangle, \quad \forall n \geq 0.$$

Multiplying both of two sides of the last inequality by 2, we obtain

$$\begin{aligned} 2\rho_n \{f(z_n, x^*) - f(z_n, t_n)\} &\geq 2\langle x_n - t_n, x^* - t_n \rangle \\ &= ||x_n - t_n||^2 + ||x^* - t_n||^2 - ||x_n - x^*||^2, \end{aligned}$$

or

$$||t_n - x^*||^2 - 2\rho_n f(z_n, x^*) \leq ||x_n - x^*||^2 - ||x_n - t_n||^2 - 2\rho_n f(z_n, t_n). \quad (3.1)$$

Using the Lipschitz-type continuous condition of f, we have

$$f(z_n, t_n) \geq f(y_n, t_n) - f(y_n, z_n) - c_1||y_n - z_n||^2 - c_2||z_n - t_n||^2$$

relation (3.1), we obtain

$$\begin{aligned} &||t_n - x^*||^2 \leq 2\rho_n f(z_n, x^*) + ||x_n - x^*||^2 - ||x_n - t_n||^2 \\ &\quad - 2\rho_n \{f(y_n, t_n) - f(y_n, z_n)\} + 2\rho_n c_1||y_n - z_n||^2 + 2\rho_n c_2||z_n - t_n||^2. \quad (3.2) \end{aligned}$$

From the definition of $z_n = prox_{\rho_n f(y_n,.)}(y_n)$ and Lemma 2 with $y = t_n$, we have

$$\begin{aligned} &2\rho_n \{f(y_n, t_n) - f(y_n, z_n)\} \geq 2\langle y_n - z_n, t_n - z_n \rangle = 2\langle y_n - x_n, t_n - z_n \rangle \\ &\quad + 2\langle x_n - z_n, t_n - z_n \rangle = 2\langle y_n - x_n, t_n - y_n \rangle + 2\langle y_n - x_n, y_n - z_n \rangle \\ &\quad + 2\langle x_n - z_n, t_n - z_n \rangle = 2\langle y_n - x_n, t_n - y_n \rangle + ||y_n - x_n||^2 + ||y_n - z_n||^2 \\ &\quad - ||x_n - z_n||^2 + ||x_n - z_n||^2 + ||t_n - z_n||^2 - ||x_n - t_n||^2 \\ &= 2\langle y_n - x_n, t_n - y_n \rangle + ||y_n - x_n||^2 + ||y_n - z_n||^2 + ||t_n - z_n||^2 - ||x_n - t_n||^2. \end{aligned}$$

This together with relation (3.2) implies that

$$\begin{aligned} &||t_n - x^*||^2 \leq 2\rho_n f(z_n, x^*) + ||x_n - x^*||^2 - ||y_n - x_n||^2 - (1 - 2\rho_n c_1) \\ &\quad \times ||y_n - z_n||^2 - (1 - 2\rho_n c_2)||z_n - t_n||^2 + 2\langle x_n - y_n, t_n - y_n \rangle. \quad (3.3) \end{aligned}$$

It follows from the definition of $y_n = prox_{\lambda_n f(x_n,.)}(x_n)$ and Lemma 2 with $y = t_n$ that

$$\lambda_n \{f(x_n, t_n) - f(x_n, y_n)\} \geq \langle x_n - y_n, t_n - y_n \rangle . \tag{3.4}$$

Thus, from relation (3.3), we obtain

$$\begin{aligned}
&||t_n - x^*||^2 \leq 2\rho_n f(z_n, x^*) + ||x_n - x^*||^2 - ||y_n - x_n||^2 - (1 - 2\rho_n c_1) \\
&\quad \times ||y_n - z_n||^2 - (1 - 2\rho_n c_2)||z_n - t_n||^2 + 2\lambda_n(f(x_n, t_n) - f(x_n, y_n)) \\
&\leq ||x_n - x^*||^2 - (1 - 6\rho_n c_1)||y_n - x_n||^2 - (1 - 2\rho_n c_1 - 3\rho_n c_2)||y_n - z_n||^2 \\
&\quad - \frac{\Delta_n}{2}||z_n - t_n||^2 + 2\lambda_n(f(x_n, t_n) - f(x_n, y_n)) \\
&= ||x_n - x^*||^2 - T_n + 2\lambda_n(f(x_n, t_n) - f(x_n, y_n)),
\end{aligned} \tag{3.5}$$

where, in the second inequality, we have added two non-negative terms $6\rho_n c_1||y_n - x_n||^2$, $3\rho_n c_2||y_n - z_n||^2$ and used the fact from the definition of Δ_n that $\frac{\Delta_n}{2} \leq \Delta_n \leq 1 - 2\rho_n c_2$.

Now, if $f(x_n, t_n) - f(x_n, y_n) \leq 0$, from relation (3.5) and $\lambda_n \geq 0$, we obtain the desired conclusion. Otherwise, assume that $f(x_n, t_n) - f(x_n, y_n) > 0$. Thus, from relation (3.4) and $\rho_n \geq \lambda_n \geq 0$, we get

$$\langle x_n - y_n, t_n - y_n \rangle \leq \lambda_n \{f(x_n, t_n) - f(x_n, y_n)\} \leq \rho_n \{f(x_n, t_n) - f(x_n, y_n)\} . \tag{3.6}$$

Next, using relation (3.1), and after that, combining with relation (3.6), we have

$$\begin{aligned}
&||t_n - x^*||^2 - 2\rho_n f(z_n, x^*) \leq ||x_n - x^*||^2 - ||x_n - t_n||^2 - 2\rho_n f(z_n, t_n) \\
&\leq ||x_n - x^*||^2 - ||(x_n - y_n) - (t_n - y_n)||^2 - 2\rho_n f(z_n, t_n) \\
&\leq ||x_n - x^*||^2 - ||x_n - y_n||^2 - ||t_n - y_n||^2 + 2\langle x_n - y_n, t_n - y_n \rangle - 2\rho_n f(z_n, t_n) \\
&\leq ||x_n - x^*||^2 - ||x_n - y_n||^2 - ||t_n - y_n||^2 \\
&\quad + 2\rho_n \{f(x_n, t_n) - f(x_n, y_n)\} - 2\rho_n f(z_n, t_n).
\end{aligned} \tag{3.7}$$

From $z_n = prox_{\rho_n f(y_n,.)}(y_n)$ and Lemma 2 with $y = y_n$, we get

$$-\rho_n f(y_n, z_n) = \rho_n(f(y_n, y_n) - f(y_n, z_n)) \geq \langle y_n - z_n, y_n - z_n \rangle = ||y_n - z_n||^2,$$

in which the first equality is followed from the fact that $f(y_n, y_n) = 0$. Thus

$$-2\rho_n f(y_n, z_n) - 2||y_n - z_n||^2 \geq 0.$$

Adding this non-negative term to the right-hand side of inequality (3.7), we obtain

$$\begin{aligned}
&||t_n - x^*||^2 - 2\rho_n f(z_n, x^*) \leq ||x_n - x^*||^2 - ||x_n - y_n||^2 - ||t_n - y_n||^2 \\
&\quad - 2||y_n - z_n||^2 + 2\rho_n [f(x_n, t_n) - f(z_n, t_n) - f(x_n, y_n) - f(y_n, z_n)] .
\end{aligned} \tag{3.8}$$

Using the Lipschitz-type continuous condition of f, we obtain

$$\begin{aligned}
&f(x_n,t_n)-f(z_n,t_n)-f(x_n,y_n)-f(y_n,z_n)\\
&\quad\le f(x_n,z_n)+c_1||x_n-z_n||^2+c_2||z_n-t_n||^2-f(x_n,y_n)-f(y_n,z_n)\\
&\quad\le c_1||x_n-z_n||^2+c_2||z_n-t_n||^2+[f(x_n,z_n)-f(x_n,y_n)-f(y_n,z_n)]\\
&\quad\le c_1||x_n-z_n||^2+c_2||z_n-t_n||^2+c_1||x_n-y_n||^2+c_2||y_n-z_n||^2. \qquad (3.9)
\end{aligned}$$

Combining relations (3.8) and (3.9), we obtain

$$\begin{aligned}
&||t_n-x^*||^2-2\rho_n f(z_n,x^*)\le ||x_n-x^*||^2-||x_n-y_n||^2-||t_n-y_n||^2\\
&\quad-2||y_n-z_n||^2+2\rho_n\left[c_1||x_n-z_n||^2+c_2||z_n-t_n||^2+c_1||x_n-y_n||^2\right.\\
&\quad\left.+c_2||y_n-z_n||^2\right]=||x_n-x^*||^2-(1-2\rho_n c_1)||x_n-y_n||^2-(2-2\rho_n c_2)\\
&\quad\times||y_n-z_n||^2+2\rho_n c_1||x_n-z_n||^2+2\rho_n c_2||z_n-t_n||^2-||t_n-y_n||^2. \quad (3.10)
\end{aligned}$$

Using the triangle inequality and the inequality $(a+b)^2\le 2(a^2+b^2)$ for all $a,\ b\in\mathcal{R}$, we obtain

$$\begin{aligned}
&||x_n-z_n||^2\le(||x_n-y_n||+||y_n-z_n||)^2\le 2(||x_n-y_n||^2+||y_n-z_n||^2),\\
&||z_n-t_n||^2\le(||z_n-y_n||+||y_n-t_n||)^2\le 2(||z_n-y_n||^2+||y_n-t_n||^2).
\end{aligned}$$

Combining relation (3.10) and the last inequalities, we obtain

$$\begin{aligned}
&||t_n-x^*||^2-2\rho_n f(z_n,x^*)\le||x_n-x^*||^2-(1-6\rho_n c_1)||x_n-y_n||^2\\
&\quad-2(1-3\rho_n c_2-2\rho_n c_1)||y_n-z_n||^2-(1-4\rho_n c_2)||t_n-y_n||^2\\
&=||x_n-x^*||^2-(1-6\rho_n c_1)||x_n-y_n||^2-(1-3\rho_n c_2-2\rho_n c_1)||y_n-z_n||^2\\
&\quad-\left[(1-3\rho_n c_2-2\rho_n c_1)||y_n-z_n||^2+(1-4\rho_n c_2)||t_n-y_n||^2\right]\\
&\le||x_n-x^*||^2-(1-6\rho_n c_1)||x_n-y_n||^2-(1-3\rho_n c_2-2\rho_n c_1)||y_n-z_n||^2\\
&\quad-\Delta_n\left[||y_n-z_n||^2+||t_n-y_n||^2\right],
\end{aligned}$$

in which the last inequality is followed from the definiton of Δ_n. Moreover, we have

$$||t_n-z_n||^2\le(||t_n-y_n||+||y_n-z_n||)^2\le 2\left(||t_n-y_n||^2+||y_n-z_n||^2\right).$$

Combining the last inequalities and the definition of T_n, we obtain the desired conclusion. This completes the proof of Lemma 5. □

Lemma 6. *Under Conditions (A1)–(A4) and (B1)–(B2), there hold the following estimates for all* $x^*\in EP(f,C)$ *and* $n\ge 0$,

(i) $||t_n-x^*||^2\le||x_n-x^*||^2-T_n$,

(ii) $||x_{n+1}-x^*||^2\le||x_n-x^*||^2-(1-\alpha_n)T_n$.

Proof. (i) Since $x^*\in EP(f,C)$, $f(x^*,z_n)\ge 0$. It follows from the pseudomonotonicity of f that $f(z_n,x^*)\le 0$. This together with Lemma 5 and $\rho_n>0$ implies conclusion (i).

(ii) It follows from the definition of x_{n+1}, the convexity of $||.||^2$ and (i) that

$$||x_{n+1} - x^*||^2 = ||\alpha_n(x_n - x^*) + (1 - \alpha_n)(t_n - x^*)||^2 \leq \alpha_n||x_n - x^*||^2 + (1 - \alpha_n)||t_n - x^*||^2 \leq \alpha_n||x_n - x^*||^2 + (1 - \alpha_n)(||x_n - x^*||^2 - T_n) = ||x_n - x^*||^2 - (1 - \alpha_n)T_n.$$

This completes the proof of Lemma 6. □

Next, we will prove the weak convergence of Algorithm 1. For obtaining this weak convergence, we need condition (A5) and two additional hypotheses on ρ_n and α_n. Precisely, we have the following result.

Theorem 1 [Weakly convergent theorem]. *Suppose that Conditions (A1)–(A5) and (B1)–(B2) hold. In addition, suppose that* $\lim_{n\to\infty}\inf \rho_n > 0$ *and* $\lim_{n\to\infty}\sup \alpha_n < 1$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 1 converges weakly to a point* $p \in EP(f, C)$. *Moreover,* $p = \lim_{n\to\infty} P_{EP(f,C)}(x_n)$.

Proof. Put $a_n = ||x_n - x^*||^2$ and $b_n = (1 - \alpha_n)T_n$. From Lemma 6 (ii), we have

$$a_{n+1} \leq a_n - b_n. \tag{3.11}$$

Thus, there exists the limit of $\{a_n\}$ and $\sum_{n=0}^{\infty} b_n < +\infty$. This implies that $\{a_n\}$, and therefore $\{x_n\}$, are bounded, and $b_n \to 0$ when $n \to \infty$. Thus, it follows from the definition of b_n and $\lim_{n\to\infty}\sup \alpha_n < 1$ that $T_n \to 0$. This together with the definitions of T_n, Δ_n and Condition (B1) implies that

$$\lim_{n\to\infty} ||y_n - x_n|| = \lim_{n\to\infty} ||y_n - z_n|| = \lim_{n\to\infty} ||z_n - t_n|| = 0. \tag{3.12}$$

Now, assume that p is some weak cluster points of $\{x_n\}$. Without loss of generality, we can write $x_n \rightharpoonup p$ as $n \to \infty$. Since C is convex and closed, C is weakly closed. Hence $p \in C$. Moreover, from (3.12), we also obtain $y_n,\ z_n,\ t_n \rightharpoonup p$ when $n \to \infty$. It follows from Lemma 2 and $t_n = prox_{\rho_n f(z_n,.)}(x_n)$ that

$$\rho_n f(z_n, y) \geq \rho_n f(z_n, t_n) + \langle x_n - t_n, y - t_n\rangle\,,\ \forall y \in C. \tag{3.13}$$

Using the Lipschitz-type continuous condition of f, we get

$$f(z_n, t_n) \geq f(y_n, t_n) - f(y_n, z_n) - c_1||y_n - z_n||^2 - c_2||z_n - t_n||^2. \tag{3.14}$$

Also, from $z_n = prox_{\rho_n f(y_n,.)}(y_n)$ and Lemma 2 with $y = t_n$, we have

$$\rho_n(f(y_n, t_n) - f(y_n, z_n)) \geq \langle y_n - z_n, t_n - z_n\rangle\,. \tag{3.15}$$

Combining relations (3.13), (3.14) and (3.15), and after that, dividing both of two sides of the obtained inequatily by $\rho_n > 0$, we get, for all $y \in C$ and $n \geq 0$,

$$f(z_n, y) \geq \frac{1}{\rho_n}\langle x_n - t_n, y - t_n\rangle + \frac{1}{\rho_n}\langle y_n - z_n, t_n - z_n\rangle - c_1||y_n - z_n||^2 - c_2||z_n - t_n||^2. \tag{3.16}$$

Thus, passing to the limit in (3.16) and using relation (3.12), $\lim_{n\to\infty} \inf \rho_n > 0$, (A5) and $z_n \rightharpoonup p$, we get $f(p,y) \geq 0$ for all $y \in C$ or $p \in EP(f,C)$. Now, using again relation (3.11) with $x^* = p$, we obtain

$$||x_{n+1} - p||^2 \leq ||x_n - p||^2 - b_n \leq ||x_n - p||^2.$$

Thus, since p is a weak cluster point of $\{x_n\}$, the whole sequence $\{x_n\}$ converges weakly to p when $n \to \infty$. Moreover $p = \lim_{n\to\infty} P_{EP(f,C)}(x_n)$. Theorem 1 is proved. □

4 Halpern multi-step proximal-like algorithm

In this section, we propose a strongly convergent Algorithm which is called the Halpern Multi-step Proximal-like Algorithm (Halpern MPA).

Algorithm 2 [Halpern MPA].
Initialization. Choose $x_0 \in C$ and parameters $\lambda_n > 0,\ \rho_n > 0,\ \alpha_n \in (0,1)$.
Step 1. Compute

$$y_n = prox_{\lambda_n f(x_n,.)}(x_n), \quad z_n = prox_{\rho_n f(y_n,.)}(y_n), \quad t_n = prox_{\rho_n f(z_n,.)}(x_n).$$

Step 2. Compute

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)t_n.$$

Set $n := n+1$ and go back **Step 1**.

Before proving the strongly convergent theorem, we need the following lemma.

Lemma 7. *Suppose that Conditions (A1)–(A4) and (B1)–(B2) hold. Then*

(i) *The sequences* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{t_n\}$ *generated by Algorithm 2 are bounded.*

(ii) *There holds the following estimate for all* $n \geq 0$ *and* $x^* \in EP(f,C)$,

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - T_n - 2\alpha_n \langle x_{n+1} - x^*, t_n - x_0\rangle - \alpha_n^2 ||t_n - x_0||^2.$$

Proof. (i) It follows from Lemma 6 (i) and (B1) that $||t_n - x^*|| \leq ||x_n - x^*||$. Thus, from the definition of x_{n+1} and the triangle inequality, we obtain

$$\begin{aligned}||x_{n+1} - x^*|| &= ||\alpha_n x_0 + (1-\alpha_n)t_n - x^*||\\ &= ||(1-\alpha_n)(t_n - x^*) + \alpha_n(x_0 - x^*)|| \leq (1-\alpha_n)||t_n - x^*||\\ &\quad + \alpha_n||x_0 - x^*|| \leq (1-\alpha_n)||x_n - x^*|| + \alpha_n||x_0 - x^*||\\ &\leq \max\{||x_n - x^*||, ||x_0 - x^*||\}.\end{aligned}$$

Therefore, by induction, we obtain

$$||x_n - x^*|| \leq ||x_0 - x^*||, \ \forall n \geq 0.$$

This implies that the sequence $\{x_n\}$, and so $\{t_n\}$, are bounded. Thus, from Lemma 6 (i), we obtain that $\{T_n\}$ is bounded. Hence, two sequences $\{y_n\}$, $\{z_n\}$

are also bounded which is followed from the definition of T_n and Condition (B1).
(ii) From the definition of x_{n+1}, we have

$$t_n - x^* = x_{n+1} - x^* + \alpha_n(t_n - x_0).$$

Thus

$$\begin{aligned}||t_n - x^*||^2 &= ||(x_{n+1} - x^*) + \alpha_n(t_n - x_0)||^2 \\ &= ||x_{n+1} - x^*||^2 + 2\alpha_n \langle x_{n+1} - x^*, t_n - x_0\rangle + \alpha_n^2||t_n - x_0||^2,\end{aligned}$$

which follows from Lemma 6 (i) that

$$\begin{aligned}||x_{n+1} - x^*||^2 &= ||t_n - x^*||^2 - 2\alpha_n \langle x_{n+1} - x^*, t_n - x_0\rangle - \alpha_n^2||t_n - x_0||^2 \\ &\leq ||x_n - x^*||^2 - T_n - 2\alpha_n \langle x_{n+1} - x^*, t_n - x_0\rangle - \alpha_n^2||t_n - x_0||^2.\end{aligned}$$

This completes the proof of Lemma 7. □

In order to establish the strong convergence of Algorithm 2, we consider condition (A5), an additional hypothesis on ρ_n that $\lim_{n\to\infty} \inf \rho_n > 0$, and the following assumptions on parameter sequence $\{\alpha_n\}$.

(B3) $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$.

Theorem 2. *Suppose that Conditions (A1)–(A5) and (B1)–(B3) hold. In addition, suppose that* $\lim_{n\to\infty} \inf \rho_n > 0$*. Then, the sequence* $\{x_n\}$ *generated by Algorithm 2 converges strongly to* $x^\dagger = P_{EP(f,C)}(x_0)$*.*

Proof. It follows from the boundedness of two sequences $\{x_n\}$ and $\{t_n\}$ that there exists $M > 0$ such that $2\left|\left\langle x_{n+1} - x^\dagger, t_n - x_0\right\rangle\right| \leq M$. Thus, from Lemma 7 (ii) with $x^* = x^\dagger$, we obtain

$$a_{n+1} - a_n + T_n + \alpha_n^2||t_n - x_0||^2 \leq \alpha_n M, \tag{4.1}$$

where $a_n = ||x_n - x^\dagger||^2$. We consider two cases.

Case 1. There exists n_0 such that $\{a_n\}$ is decreasing for all $n \geq n_0$. Thus, there exists the limit of $\{a_n\}$, i.e., $\lim_{n\to\infty} ||x_n - x^\dagger||^2 = a$ and $a_{n+1} - a_n \to 0$. It follows from (4.1) and $\alpha_n \to 0$ that $T_n \to 0$. Since Condition (B1) and the definition of T_n, we get

$$||x_n - y_n|| \to 0,\ ||y_n - z_n|| \to 0,\ ||t_n - z_n||^2 \to 0,\ \alpha_n^2||t_n - x_0||^2 \to 0. \tag{4.2}$$

Therefore, it follows from the triangle inequality that $||x_n - t_n|| \to 0$, and then, $||t_n - x^\dagger||^2 \to a$ because $\lim_{n\to\infty} ||x_n - x^\dagger||^2 = a$. From relation (4.2) and the definition of x_{n+1},

$$||x_{n+1} - t_n||^2 = \alpha_n^2||z_n - x_0||^2 \to 0. \tag{4.3}$$

Note that $\{t_n\}$ is bounded. Without loss of generality, we can assume that there exists a subsequence $\{t_m\}$ of $\{t_n\}$ converging weakly to p such that

$$\lim_{n\to\infty} \inf \left\langle t_n - x^\dagger, x^\dagger - x_0\right\rangle = \lim_{m\to\infty} \left\langle t_m - x^\dagger, x^\dagger - x_0\right\rangle. \tag{4.4}$$

Since C is closed and convex, C is weakly closed. Thus, from $t_m \rightharpoonup p$ and $\{t_m\} \subset C$, we obtain $p \in C$. Moreover, from (4.2), we also obtain $z_m \rightharpoonup p$. By arguing similarly to (3.13)–(3.16), we obtain

$$\begin{aligned} f(z_m, y) \quad \geq \quad & \frac{1}{\rho_m}\langle x_m - t_m, y - t_m\rangle + \frac{1}{\rho_m}\langle y_m - z_m, t_m - z_m\rangle \\ & -c_1||y_m - z_m||^2 - c_2||z_m - t_m||^2. \end{aligned}$$

Thus, passing to the limit in the last inequality and using relation (4.2), $\lim_{n\to\infty}\inf \rho_n > 0$ and (A5), we get $f(p, y) \geq 0$ for all $y \in C$ or $p \in EP(f, C)$. From (4.4), $x^\dagger = P_{EP(f,C)}(x_0)$ and Lemma 1 (ii), one has

$$\lim_{n\to\infty}\inf \left\langle t_n - x^\dagger, x^\dagger - x_0\right\rangle = \lim_{m\to\infty}\left\langle t_m - x^\dagger, x^\dagger - x_0\right\rangle = \left\langle p - x^\dagger, x^\dagger - x_0\right\rangle \geq 0. \tag{4.5}$$

We have the following fact

$$\begin{aligned} \left\langle x_{n+1} - x^\dagger, t_n - x_0\right\rangle &= \langle x_{n+1} - t_n, t_n - x_0\rangle + \left\langle t_n - x^\dagger, t_n - x_0\right\rangle \\ &= \langle x_{n+1} - t_n, t_n - x_0\rangle + \left\langle t_n - x^\dagger, t_n - x^\dagger\right\rangle + \left\langle t_n - x^\dagger, x^\dagger - x_0\right\rangle \\ &= \langle x_{n+1} - t_n, t_n - x_0\rangle + ||t_n - x^\dagger||^2 + \left\langle t_n - x^\dagger, x^\dagger - x_0\right\rangle. \end{aligned}$$

Combining this equality with relations (4.3), (4.5) and $\lim_{n\to\infty} ||t_n - x^\dagger||^2 = a$, we obtain

$$\lim_{n\to\infty}\inf \left\langle x_{n+1} - x^\dagger, z_n - x_0\right\rangle \geq a.$$

Assume that $a > 0$ then, from the last inequality, there exists $n_0 \geq 0$ such that

$$\left\langle x_{n+1} - x^\dagger, t_n - x_0\right\rangle \geq a/2,\ \forall n \geq n_0.$$

Therefore, from Lemma 7 (ii),

$$a_{n+1} \leq a_n - a\alpha_n,\ \forall n \geq n_0.$$

Thus, for all $\forall n \geq n_0$,

$$a_{n+1} \leq a_{n_0} - a\sum_{i=n_0}^{n} \alpha_i.$$

The last inequality together with the hypothesis $\sum_{i=1}^{\infty} \alpha_i = +\infty$ and $a > 0$ implies that $a_{n+1} \to -\infty$. This is contrary. So, $a = 0$ and $||x_n - x^\dagger|| \to 0$.

Case 2. There exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} \leq a_{n_j+1}$ for all $j \geq 0$. From Lemma 4,

$$a_{\tau(n)} \leq a_{\tau(n)+1},\ a_n \leq a_{\tau(n)+1},\ \forall n \geq n_0, \tag{4.6}$$

where $\tau(n) = \max\{k \in N : n_0 \leq k \leq n,\ a_k \leq a_{k+1}\}$. Thus, from relation (4.1), $a_{\tau(n)} \leq a_{\tau(n)+1}$ and $\alpha_n \to 0$, we obtain $T_{\tau(n)} \to 0$ and $\alpha^2_{\tau(n)}||t_{\tau(n)} - x_0||^2 \to 0$. Thus, from the definition of $T_{\tau(n)}$ and Condition (B1), we get

$$\begin{aligned} &||y_{\tau(n)} - x_{\tau(n)}||^2 \to 0,\ ||z_{\tau(n)} - y_{\tau(n)}||^2 \to 0, \\ &||t_{\tau(n)} - z_{\tau(n)}||^2 \to 0,\ \alpha^2_{\tau(n)}||t_{\tau(n)} - x_0||^2 \to 0. \end{aligned} \tag{4.7}$$

From the definition of x_n, we obtain

$$||x_{\tau(n)+1} - t_{\tau(n)}||^2 = \alpha^2_{\tau(n)}||t_{\tau(n)} - x_0||^2 \to 0. \tag{4.8}$$

Note that $\{x_{\tau(n)}\}$ is bounded, without loss of generality, we can assume that there exists a subsequence $\{x_{\tau(n_j)}\}$ of $x_{\tau(n)}$ converging weakly to $p \in C$ as $j \to \infty$ such that

$$\begin{aligned}\lim_{n\to\infty} \sup \langle x_{\tau(n)+1} - x^\dagger, x_0 - x^\dagger\rangle &= \lim_{j\to\infty} \langle x_{\tau(n_j)+1} - x^\dagger, x_0 - x^\dagger\rangle \\ &= \langle p - x^\dagger, x_0 - x^\dagger\rangle. \end{aligned} \tag{4.9}$$

From (4.7), we also have $z_{\tau(n)} \rightharpoonup p$. By arguing similarly to Case 1, we also obtain $p \in EP(f, C)$. This together with Lemma 1 (ii) and $x^\dagger = P_{EP(f,C)}$ implies that $\langle p - x^\dagger, x_0 - x^\dagger\rangle \leq 0$. Thus, from (4.9) we obtain

$$\lim_{n\to\infty} \sup \langle x_{\tau(n)+1} - x^\dagger, x_0 - x^\dagger\rangle \leq 0.$$

Now, we will show that $t_{\tau(n)} \to x^\dagger = P_{EP(f,C)}(x_0)$. From Lemma 7 (ii) and $a_{\tau(n)} \leq a_{\tau(n)+1}$,

$$2\alpha_{\tau(n)} \langle x_{\tau(n)+1} - x^\dagger, t_{\tau(n)} - x_0\rangle \leq 0.$$

Thus, $\langle x_{\tau(n)+1} - x^\dagger, t_{\tau(n)} - x_0\rangle \leq 0$, which implies that

$$\begin{aligned}||t_{\tau(n)} - x^\dagger||^2 &= \langle t_{\tau(n)} - x^\dagger, t_{\tau(n)} - x^\dagger\rangle = \langle x_{\tau(n)+1} - x^\dagger, t_{\tau(n)} - x^\dagger\rangle \\ &- \langle x_{\tau(n)+1} - t_{\tau(n)}, t_{\tau(n)} - x^\dagger\rangle = \langle x_{\tau(n)+1} - x^\dagger, t_{\tau(n)} - x_0\rangle \\ &+ \langle x_{\tau(n)+1} - x^\dagger, x_0 - x^\dagger\rangle - \langle x_{\tau(n)+1} - t_{\tau(n)}, t_{\tau(n)} - x^\dagger\rangle \\ &\leq \langle x_{\tau(n)+1} - x^\dagger, x_0 - x^\dagger\rangle - \langle x_{\tau(n)+1} - t_{\tau(n)}, t_{\tau(n)} - x^\dagger\rangle.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \sup ||t_{\tau(n)} - x^\dagger||^2 \leq \lim_{n\to\infty} \sup \langle x_{\tau(n)+1} - x^\dagger, x_0 - x^\dagger\rangle \leq 0,$$

where we have used relation (4.8). Thus, $\lim_{n\to\infty} \sup ||t_{\tau(n)} - x^\dagger||^2 = 0$, and so $\lim_{n\to\infty} ||t_{\tau(n)} - x^\dagger||^2 = 0$. Therefore, it follows from relation (4.8) that $\lim_{n\to\infty} ||x_{\tau(n)+1} - x^\dagger||^2 = 0$ or $a_{\tau(n)+1} \to 0$. Thus, from (4.6), we obtain $a_n \leq a_{\tau(n)+1} \to 0$. Hence, $x_n \to x^\dagger$ as $n \to \infty$. □

5 MPA for strongly pseudomonotone EPs

In this section, we consider a special class of strongly pseudomonotone and Lipschitz-type continuous bifunctions. Algorithms 1 and 2 can be used to solve EPs for this class of bifunctions. However, in order to construct solution approximation sequences, we need to know the information of two Lipschitz-type continuous constants c_1 and c_2. In some cases, these constants can be not easy to approximate. The presented algorithm below is for this purpose. The construction of iterative sequences and the proof of their convergence do not require the prior knowledge of the modulus of strong pseudomonotonicity and two Lipschitz-type continuous constants of bifunctions. Instead of that, we consider two control parameter sequences $\{\lambda_n\}$ and $\{\rho_n\}$ satisfying the following conditions:

(C1) $0 \le \lambda_n \le \rho_n$, (C2) $\rho_n \to 0$, $\sum_{n=1}^{\infty} \rho_n = +\infty$.

Now, we are in a position to describe the algorithm in details.

Algorithm 3 [MPA for strongly pseudomonotone EPs].
Initialization. Choose $x_0 \in C$ and parameters λ_n, ρ_n such that Conditions (C1) and (C2) above hold.
Iterative step. Compute

$$y_n = prox_{\lambda_n f(x_n,.)}(x_n), \quad z_n = prox_{\rho_n f(y_n,.)}(y_n), \quad x_{n+1} = prox_{\rho_n f(z_n,.)}(x_n).$$

Set $n := n + 1$ and go back to **Iterative step**.

For proving the convergence of Algorithm 3, we assume that bifunction f satisfies the following conditions:

(D1) $f(x, x) = 0$ for all $x \in C$ and f is strongly pseudomonotone on C;

(D2) f satisfies Lipschitz-type continuous condition on C;

(D3) $f(x, .)$ is convex and subdifferentiable on C for every fixed $x \in C$.

Note that, if $f(x, .)$ is lower semicontinuous, convex (not necessarily subdifferentiable), and $f(., y)$ is hemicontinuous on C, then under assumption (D1), EP (1.1) has an unique solution. We obtain the following strongly convergent theorem.

Theorem 3. *Suppose that Conditions (C1), (C2), (D1)–(D3) hold, and EP (1.1) has a unique solution x^*. Then $\{x_n\}$ generated by Algorithm 3 converges strongly to x^*. Moreover, if $\gamma > 0$ is the modulus of strong pseudomononicity of f, there exists a number $n_0 > 0$ such that $\gamma\rho_n < 1$ for all $n \ge n_0$ and*

$$||x_{n+1} - x^*|| \le e^{-\frac{\gamma}{3}\sum_{k=n_0}^{n}\rho_k} ||x_{n_0} - x^*||.$$

In addition, $\lim_{n\to\infty} \prod_{k=n_0}^{n} (1 - \frac{2\gamma\rho_k}{3}) = 0$.

Proof. Suppose that c_1, c_2 are two Lipschitz-type continuous constants of f. Since $\rho_n \to 0$, there exists n_0 such that, for all $n \ge n_0$,

$$\rho_n < \min\{1/6c_1, 1/4c_2, 1/(2c_1 + 3c_2)\}, \tag{5.1}$$

$$1 - 6\rho_n c_1 \ge 2\rho_n\gamma, \tag{5.2}$$

$$1 - 2\rho_n c_1 - 3\rho_n c_2 \ge 2\rho_n\gamma. \tag{5.3}$$

From relation (5.1), using Lemma 5, the definition of T_n, and the definition of x_{n+1} in Algorithm 3, we obtain, for all $n \ge n_0$,

$$\begin{aligned} ||x_{n+1} - x^*||^2 - 2\rho_n f(z_n, x^*) \le ||x_n - x^*||^2 - (1 - 6\rho_n c_1)||y_n - x_n||^2 \\ - (1 - 2\rho_n c_1 - 3\rho_n c_2)||y_n - z_n||^2. \end{aligned} \tag{5.4}$$

Since $x^* \in EP(f,C)$, $f(x^*,z_n) \geq 0$. It follows from the strong pseudomonotonicity of f that $f(z_n,x^*) \leq -\gamma||z_n - x^*||^2$. This together with inequality (5.4) implies that, for all $n \geq n_0$,

$$||x_{n+1} - x^*||^2 + 2\gamma\rho_n||z_n - x^*||^2 \leq ||x_n - x^*||^2 - (1 - 6\rho_n c_1)||y_n - x_n||^2 \\ - (1 - 2\rho_n c_1 - 3\rho_n c_2)||y_n - z_n||^2.$$

Thus, from relations (5.2) and (5.3), we obtain

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - (1 - 6\rho_n c_1)||y_n - x_n||^2 - (1 - 2\rho_n c_1 - 3\rho_n c_2) \\ \times ||y_n - z_n||^2 - 2\gamma\rho_n||z_n - x^*||^2 \leq ||x_n - x^*||^2 \\ - 2\gamma\rho_n\left[||x_n - y_n||^2 + ||y_n - z_n||^2 + ||z_n - x^*||^2\right]. \quad (5.5)$$

Using the inequality $3(a^2+b^2+c^2) \geq (a+b+c)^2$ for all $a,\ b,\ c \in \Re$, we obtain

$$||x_n - x^*||^2 \leq (||x_n - y_n|| + ||y_n - z_n|| + ||z_n - x^*||)^2 \\ \leq 3(||x_n - y_n||^2 + ||y_n - z_n||^2 + ||z_n - x^*||^2).$$

This together with relation (5.5) implies that

$$||x_{n+1}-x^*||^2 \leq ||x_n-x^*||^2 - \frac{2\rho_n\gamma}{3}||x_n-x^*||^2 = \left(1-\frac{2\rho_n\gamma}{3}\right)||x_n-x^*||^2,\ \forall n \geq n_0.$$

Note that from relation (5.2) and (C1), we obtain $2\rho_n\gamma < 1$, and so $\frac{2\rho_n\gamma}{3} \in (0,1)$. Thus, from the last inequality and the induction, we obtain, for all $n \geq n_0$,

$$||x_{n+1} - x^*||^2 \leq \prod_{k=n_0}^{n}\left(1 - \frac{2\rho_k\gamma}{3}\right)||x_{n_0} - x^*||^2. \quad (5.6)$$

Using the inequality $1 - x < e^{-x}$ for all $x > 0$, we have $1 - \frac{2\rho_k\gamma}{3} < e^{\frac{-2\gamma}{3}\rho_k}$ and hence (3) is proved. From hypothesis (C2), we also have $\frac{2\rho_n\gamma}{3} \to 0$ as $n \to \infty$ and $\sum_{n=n_0}^{\infty}\frac{2\rho_n\gamma}{3} = +\infty$. Thus, from relation (5.6) and Lemma 3, we obtain $\prod_{k=n_0}^{n}\left(1 - \frac{2\rho_k\gamma}{3}\right) \to 0$, and so $x_n \to x^*$ when $n \to \infty$. □

Remark 1. Although the construction of solution approximation sequences and obtaining their convergence do not require the prior knowledge of the modulus of strong pseudomonotonicity and two Lipschitz-type continuous constants of f, but the rate of the convergence of Algorithm 3 depends completely on these constants.

6 Computational experiments

In this section, we consider several numerical experiments to illustrate the convergence of the proposed algorithms and compare them with others. Throughout this section, we consider the following test problem introduced in [24] and later [1,4].

Problem. Let $C \subset \Re^m$ and $f : C \times C \to \Re$ be a bifunction. Find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C, \tag{6.1}$$

where $f(x,y) = \langle Px + Qy + q, y - x\rangle$,

$$C = \{x = (x_1, \ldots, x_m) \in \Re^m : x_i \geq 0, \ i = 1, \ldots, m\},$$

q is a vector in $\Re^m$ and $P, \ Q \in \Re^{m\times m}$ are two matrices of order m.

All the proximal mappings (optimization subproblems) can be solved effectively by the function *quadprog* in Matlab 7.0 Optimization Toolbox. All the programs are performed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50GHz 2.50 GHz, RAM 2.00 GB. We will compare execution time in second (CPU(s)) and number of iterations (Iter.) for mentioned algorithms.

Experiment 1. In this experiment, we compare Algorithm 1 with the extragradient method (EGM) in [24, Algorithm 1]. The vector q is generated randomly and uniformly with its entries in $[-m, m]$. Two matrices P and Q are generated randomly[1] such that Q is symmetric positive semidefinite and $Q - P$ is *negative semidefinite.* Thus, f is pseudomonotone and satisfies the Lipschitz-type continuous condition with $c_1 = c_2 = ||P - Q||/2$. We chose $\rho_n = \rho := 1/8c_1, \ \lambda_n = 0.4\rho_n, \ \alpha_n = \frac{1}{n+1}$ for the mentioned algorihms, two starting points $x_0 = (1) := (1, \ldots, 1)^T$ and $x_0 = (0) := (0, \ldots, 0)^T$. The stopping criterion is $||x_{n+1} - x_n|| \leq$ TOL. The numerical results are reported in Table 1.

Table 1. The results for Algorithm 1 and EGM in Experiment 1

		$x_0 = (1)$				$x_0 = (0)$			
		Algorithm 1		EGM		Algorithm 1		EGM	
m	TOL	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.
2	10^{-4}	0.76	19	0.72	27	0.76	17	0.64	25
	10^{-6}	1.21	29	1.22	45	1.23	27	1.28	43
5	10^{-4}	1.09	27	1.31	44	1.02	24	1.18	38
	10^{-6}	1.76	40	1.95	72	1.73	37	1.87	66
10	10^{-4}	2.01	44	2.32	80	1.99	42	2.59	76
	10^{-6}	3.14	68	4.01	132	3.15	66	3.82	128
15	10^{-4}	2.63	53	3.46	97	2.97	56	3.48	103
	10^{-6}	4.55	88	5.66	176	4.49	89	6.34	177

Experiment 2. In this experiment, we compare Algorithm 2 with the strongly convergent viscosity extragradient method (VEGM) in [25, Algorithm 1] (for $F = x - x_0$ and $S = I$). The vector q and two matrices $P, \ Q$ are generated as in Experiment 1. We chose $\rho_n = 1/6.01c_1, \ \lambda_n = 0.8\rho_n$ for Algorithm 2 and

[1] We chose two diagonal matrices Q_1, Q_2 with their diagonal elements generated randomly in $[1, m]$ and $[-m, 0]$, respectively. Next, we made the symmetric positive semidefinite matrix Q by using Q_1 and a random orthogonal matrix. Finally, we made a negative semidefinite T from Q_2 and another random orthogonal matrix, and set $P = Q - T$.

$\lambda_n = 1/2.01c_1$ for VEGM. Two parameter sequences $\{\alpha_n\}$ are $\alpha_n = \frac{1}{n+1}$ and $\alpha_n = \frac{1}{(n+1)^{0.5}}$. The stopping criterion is $||x_{n+1} - x_n|| \leq 10^{-4}$. Table 2 shows the results in this case.

Table 2. The results for Algorithm 2 and VEGM in Experiment 2

		$\alpha_n = \frac{1}{n+1}$				$\alpha_n = \frac{1}{(n+1)^{0.5}}$			
		Algorithm 2		VEGM		Algorithm 2		VEGM	
m	x_0	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.
2	(0)	1.59	17	3.19	54	1.79	21	6.92	121
	(1)	1.63	19	7.47	124	2.10	25	22.32	382
5	(0)	2.41	29	5.67	115	2.90	36	16.72	329
	(1)	2.40	30	9.17	173	3.01	37	30.87	579
10	(0)	1.77	22	8.62	80	2.29	28	19.87	204
	(1)	2.09	26	18.84	191	2.69	32	67.03	680
15	(0)	3.77	37	14.85	128	4.72	44	45.34	375
	(1)	4.33	42	25.47	227	5.21	50	93.33	847

From Experiments 1 and 2, we see that Algorithms 1 and 2 are respectively better than the classical extragradient-like algorithms in both execution time and number of iterations. In the next experiment, we study the numerical behavior of Algorithm 3 on test problem (6.1) for different choice of λ_n and ρ_n.

Experiment 3. All the entries of q, P, Q are generated randomly as Experiment 1 in $\Re^{20}$. However, unlike in Experiment 1, two matrices P, Q are generated such that the matrix $Q - P$ is *symmetric negative definite*. In this case, f also stisfies the Lipschitz-type continuous condition. Moreover, by the property of $Q - P$ and $f(x, y) \geq 0$, we have

$$\begin{aligned} f(y,x) &\leq f(y,x) + f(x,y) = \langle Py + Qx + q, x - y\rangle + \langle Px + Qy + q, y - x\rangle \\ &= \langle (P-Q)y + (Q-P)x, x - y\rangle = (x-y)^T(Q-P)(x-y) \\ &\leq -\gamma||x-y||^2, \end{aligned}$$

where some $\gamma > 0$. Thus, f is strongly pseudomonotone. We study here the numerical behavior of Algorithm 3 for different choice of ρ_n and λ_n. We have chosen $\lambda_n = 0.5\rho_n$, $\rho_n = (n+1)^{-p}$ with $p \in \{1;\ 0.75;\ 0.5;\ 0.1;\ 0.01\}$ and the starting point $x_0 = (1, \ldots, 1)^T \in \Re^{20}$. Figure 1 describes the behavior of $D_n = ||x_{n+1} - x_n||^2$ in the first 100 iterations. From this figure, we see that the convergence of D_n with $\rho_n = \frac{1}{(n+1)^p}$, $p \in \{1;\ 0.75;\ 0.5\}$ is better than others.

7 Conclusions

The paper has proposed and analyzed the convergence of three mul-step proximal-like algorithms (MPA) for solving pseudomonotone and Lipschitz-type continuous equilibrium problems in $\mathcal{H}$. The first algorithm which combines the MPA with Mann-like iteration, so-called Mann MPA, is weakly convergent while the second one, called Halpern MPA, is strongly convergent thanks

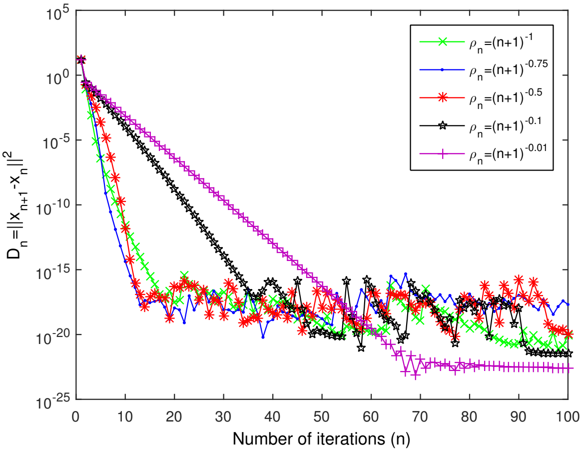

Figure 1. The numerical behavior of $D_n = ||x_{n+1} - x_n||^2$ for Algorithm 3 with different choice of ρ_n and λ_n.

to Halpern iteration. The last algorithm is designed for the special class of strongly pseudomonotone and Lipschitz-type continuous bifunctions. The main advantage of this algorithm is that the construction of solution approximation sequences and the establishing of their convergence do not requrire the prior knowledge of the modulus of strong pseudomononicity and two Lipschitz-type continuous constants of bifuntions. Finally, several preliminary numerical experiments have been performed on a test problem to illustrate the convergence of the proposed algorithms and compare them with others. The proposed algorithms have a greater number of calculations at the one iteration in comparison with classical algorithms, and this affects the computation time of one iteration (this is clearly seen in Table 1 when m=2, TOL=0.0001). But the proposed algorithms benefit significantly by the number of iterations, this affects the gain results on time.

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