$W^{1,p(x)}_0$ Versus $C^1$ Local Minimizers for Nonsmooth Functionals$^*$

Chengfu Che$^a$, Bin Ge$^b$, Xiao-Ping Xue$^a$ and Qing-Mei Zhou$^c$

$^a$Harbin Institute of Technology
150001 Harbin, China

$^b$Harbin Engineering University
150001 Harbin, China

$^c$Northeast Forestry University
150040 Harbin, China

E-mail(corresp.): hitcf0163.com
E-mail: gebin04523080261@163.com
E-mail: xiaopinxue263.net; zhouqingmei2008@163.com

Received May 9, 2011; revised February 22, 2012; published online June 1, 2012

Abstract. In this paper, we study that the local minimizers of a class of functionals (not necessarily differentiable) in the $C^1$-topology are still their local minimizers in $W^{1,p(x)}_0$.

Keywords: $p(x)$-Laplacian, Clarke’s generalized gradient, local minimizers, nonsmooth functionals.

AMS Subject Classification: 34A60; 35R70; 49J52; 49J53.

1 Introduction

We consider the functional $\varphi : W^{1,p(x)}_0(\Omega) \to \mathbb{R}$ defined by

$$
\varphi(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \int_{\Omega} F(x,u) dx,
$$

where $\Omega$ is bounded smooth domain in $\mathbb{R}^N$,

$$1 < p_- = \min_{x \in \Omega} p(x) \leq p(x) \leq \max_{x \in \Omega} p(x) = p^+ < +\infty$$

$^*$ Supported by the National Science Foundation of China (No. 11126286, 10971043, 11001063), the Fundamental Research Funds for the Central Universities (No. HEUCF 20111134), China Postdoctoral Science Foundation Funded Project (No. 20110491032) and the Natural Science Foundation of Heilongjiang Province (No. A200803).
and $F(t,x)$ is locally Lipschitz function in the $t$-variable integrand (in general it can be nonsmooth), and $\partial F(x,t)$ is defined in the sense of Clarke [3].

Such a result was first proven for functionals corresponding to elliptic equations with Dirichlet boundary values by Fan in [5]. They considered potentials of the form

$$\varphi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx - \int_\Omega F(x,u) \, dx,$$

where $F(x,u) = \int_0^u f(x,t) \, dt$ with some Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$. For variable exponent Neumann Sobolev spaces (i.e. $W_0^{1,p(x)}(\Omega)$), the result can be found in Gasinski–Papageorgiou [7]. For the case $p(x) \equiv p = 2$, Brezis and Nirenberg in [2] proved that the local minimizers of a class of functionals in the $C^1$-topology are still their local minimizers in $W_0^{1,p}(\Omega)$. For the case when $p(x) \equiv p$, Motreanu and Papageorgiou [9] first considered the problem by using nonsmooth functionals defined on $W_0^{1,p}(\Omega)$ for the case $2 \leq p < +\infty$. In Iannizzotto–Papageorgiou [8] an analogous result was proved for $p > 1$, Neumann boundary conditions and a nonsmooth potential.

The idea of the present paper was the generalization to the more general case of nonsmooth functionals defined on $W_0^{1,p(x)}(\Omega)$ with $1 < p^- \leq p^+ < +\infty$. The main goal of this paper is to answer the following question:

(Q) If $u_0 \in W_0^{1,p(x)}(\Omega)$ is a local minimizer of $\varphi$ in the $C^1$-topology, is it still a local minimizer of $\varphi$ in $W_0^{1,p(x)}(\Omega)$?

2 Hypotheses

We suppose the following conditions on the nonsmooth potentials $F : \Omega \times \mathbb{R} \to \mathbb{R}$:

$H(F)$:

1. For all $t \in \mathbb{R}$, $x \mapsto F(x,t)$ is measurable;

2. For almost all $x \in \mathbb{R}^N$, $t \mapsto F(x,t)$ is locally Lipschitz;

3. For almost all $x \in \Omega$, all $t \in \mathbb{R}$ and $w \in \partial F(x,t)$, we have $|w| \leq a_1 + a_2 |t|^\alpha(x)-1$, where $a_1$, $a_2$ are positive constants, $\alpha \in C(\overline{\Omega})$ with $1 < \alpha^- \leq \alpha(x) < p^*(x)$, $p^*(x) = \frac{Np(x)}{N-p(x)}(+\infty)$, if $p(x) < N(p(x) \geq N)$.

$H(p)$: $p \in C^{0,\beta}(\overline{\Omega})$ with $\beta \in (0,1)$, $1 < p(x) < p^*(x)$ for every $x \in \overline{\Omega}$.

Remark 1. Note that the conditions above imply that the functional $\varphi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is locally Lipschitz (similar to the proof of Lemma 3.1 in [6]). This guarantees, in particular, that Clarke’s generalized gradient $t \mapsto \partial \varphi(t)$ exists.

3 Sobolev Versus $C^1$ Local Minimizers

In this section we shall give a positive answer to our question (Q). Our main result is stated in the following theorem.

Theorem 1. Suppose that $H(F)$ and $H(p)$ hold, and that $u_0 \in W^{1,p(x)}_0(\Omega)$ is a local minimizer of $\varphi$ in the $C_0^1(\overline{\Omega})$ topology, that is, there exists some $r > 0$ such that

$$\varphi(u_0) \leq \varphi(u_0 + h) \quad \text{for all } h \in C_0^1(\overline{\Omega}) \text{ with } \|h\|_{C_0^1(\overline{\Omega})} \leq r,$$

then $u_0 \in C_0^1(\overline{\Omega})$ is a local minimizer of $\varphi$ in the $W^{1,p(x)}_0(\Omega)$ topology, i.e., there exists $k > 0$ such that

$$\varphi(u_0) \leq \varphi(u_0 + h) \quad \text{for all } h \in W^{1,p(x)}_0(\Omega) \text{ with } \|h\|_{W^{1,p(x)}_0(\Omega)} \leq k.$$

Proof. By assumption, there exists $r_0 > 0$ such that

$$\varphi(u_0) \leq \varphi(u_0 + h), \quad \text{for all } h \in C_0^1(\overline{\Omega}), \quad \|h\|_{C_0^1(\overline{\Omega})} \leq r_0.$$

Choose $h \in C_0^1(\overline{\Omega})$, then for small enough $t > 0$ we have $\varphi(u_0) \leq \varphi(u_0 + th)$, hence,

$$0 \leq \varphi^\circ(u_0; h). \quad (3.1)$$

Since $C_0^1(\overline{\Omega})$ is dense in $W^{1,p(x)}_0(\Omega)$ and $\varphi^\circ$ is continuous, (3.1) holds for all $h \in W^{1,p(x)}_0(\Omega)$, so $0 \in \partial \varphi(u_0)$. For any $u \in W^{1,p(x)}_0(\Omega)$, define

$$\beta(x) = \frac{\alpha(x)}{\alpha(x) - 1},$$

$$N(u) = \{ w \in L^{\beta(x)}(\Omega); w(x) \in \partial F(x, u(x)) \text{ for a.e. } x \in \Omega \}.$$

Then, there exists $v_0 \in N(u_0)$ such that

$$-\operatorname{div}(|\nabla u_0|^{p(x)} - 2\nabla u_0) = v_0. \quad (3.2)$$

From (3.2), we infer that

$$\begin{cases}
-\operatorname{div}(|\nabla u_0(x)|^{p(x)} - 2\nabla u_0(x)) = v_0(x), & \text{a.e. } x \in \Omega, \\
0_{\partial\Omega} = 0. &
\end{cases} \quad (3.3)$$

From (3.3) and nonlinear regularity (see Fan and Zhao [6, Theorem 4.1] and Fan [4, Theorem 1.2]), we have $u_0 \in C_0^1(\overline{\Omega})$, for some $\alpha \in (0, 1)$.

Now, we argue by contradiction. Suppose that $u_0$ is not a local minimizer of $\varphi$ in the $W^{1,p(x)}_0(\Omega)$. Exploiting the compact embedding of $W^{1,p(x)}_0(\Omega)$ into $L^{p(x)}(\Omega)$, we can easily check that $\varphi$ is sequentially weakly lower semicontinuous.

Define $G(u) = \int_{\Omega} \frac{|\nabla u - \nabla u_0|^{p(x)}}{p(x)} \, dx$, $\forall u \in W^{1,p(x)}_0(\Omega)$.

Fix $\varepsilon \in (0, 1]$, see $\overline{B}_\varepsilon = \{ u \in W^{1,p(x)}_0(\Omega); G(u) \leq \varepsilon \}$. Then $\overline{B}_\varepsilon$ is weakly compact, closet convex subset of $W^{1,p(x)}_0(\Omega)$ and is a neighbourhood of $u_0$ in $W^{1,p(x)}_0(\Omega)$. From the Weierstrass theorem we can find $u_\varepsilon \in \overline{B}_\varepsilon$ such that

$$\varphi(u_\varepsilon) = \inf_{u \in \overline{B}_\varepsilon} \varphi(u) < \varphi(u_0).$$
By the nonsmooth multiplier rule of Clarke [3] (Theorem 1 and Proposition 13), we can find $\lambda_\varepsilon \leq 0$ such that $\lambda_\varepsilon G'(u_\varepsilon) \in \partial \varphi(u_\varepsilon)$. Hence,

$$A(u_\varepsilon) - w_\varepsilon = \lambda_\varepsilon A(u_\varepsilon - u_0), \quad w_\varepsilon \in N(u_\varepsilon).$$

That is,

$$-\text{div}\{\nabla u_\varepsilon |p(x)-2\nabla u_\varepsilon - \lambda_\varepsilon \nabla u_\varepsilon - \nabla u_0|p(x)-2(\nabla u_\varepsilon - \nabla u_0)\} = w_\varepsilon. \quad (3.4)$$

Define $A_\varepsilon : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $B_\varepsilon : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$A_\varepsilon(x, \eta) = \frac{1}{1 - \lambda_\varepsilon} |\eta|^{p(x)-2} \eta - \frac{\lambda_\varepsilon}{1 - \lambda_\varepsilon} |\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0),$$

$$B_\varepsilon(x, \eta) = \frac{1}{1 - \lambda_\varepsilon} w_\varepsilon,$$

where $w_\varepsilon \in \partial F(x, \eta)$. Then $u_\varepsilon$ is a solution of the following problem:

$$\begin{cases}
-\text{div}A_\varepsilon(x, \nabla u) = B_\varepsilon(x, u), & \text{a.e. } x \in \Omega, \\
u|_{\partial \Omega} = 0.
\end{cases} \quad (3.5)$$

We can verify that $A_\varepsilon$ and $B_\varepsilon$ satisfy the following conditions:

$$A_\varepsilon(x, \eta) \eta \geq c_1 |\eta|^{p(x)} - c_2, \quad \forall x \in \overline{\Omega}, \quad \eta \in \mathbb{R}^N, \quad (3.6)$$

$$|A_\varepsilon(x, \eta)| \leq c_3 |\eta|^{p(x)-1} + c_4, \quad \forall x \in \overline{\Omega}, \quad \eta \in \mathbb{R}^N, \quad (3.7)$$

$$|B_\varepsilon(x, \eta)| \leq c_5 + c_6 |\eta|^{\alpha(x)-1}, \quad \forall x \in \overline{\Omega}, \quad \eta \in \mathbb{R}^N, \quad (3.8)$$

where $c_i$ are positive constants independent of $\varepsilon \in (0, 1)$.

Since the verification of (3.7) and (3.8) is simple, we only give the proof of (3.6) here. By the definition of $A_\varepsilon(x, \eta)$,

$$A_\varepsilon(x, \eta) \eta = \frac{1}{1 - \lambda_\varepsilon} \left[ (|\eta|^{p(x)-2}\eta - \lambda_\varepsilon |\eta|^{p(x)-2}\eta) - \lambda_\varepsilon \left(|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - |\eta|^{p(x)-2}\eta\right) \right] \eta$$

$$= \frac{1}{1 - \lambda_\varepsilon} \left[ (1 - \lambda_\varepsilon)(|\eta^{p(x)} - \lambda_\varepsilon I|) \right],$$

where $I = (|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - |\eta|^{p(x)-2}\eta)\eta$.

Note that,

$$|\xi|^{p-2} - |\eta|^{p-2} \eta \leq \begin{cases}
c_0 |\xi - \eta|(|\xi| + |\eta|)^{p-2}, & \text{if } p \geq 2, \\
c_0 |\xi - \eta|^{p-1}, & \text{if } 1 < p < 2,
\end{cases} \quad \text{see Azorero [1].}$$

So, when $p(x) \geq 2$, we have

$$|I| = \left| (|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - |\eta|^{p(x)-2}\eta) \right| |\eta|$$

$$\leq c_0 |\nabla u_0|( |\eta - \nabla u_0| + |\eta|)^{p(x)-2} |\eta|$$

$$\leq C |\nabla u_0|(|\eta|^{p(x)-2} + |\nabla u_0|^{p(x)-2}) |\eta|$$

$$\leq C |\eta|^{p(x)-1} + C \leq \frac{1}{2} |\eta|^{p(x)} + C,$$
and when \( p(x) < 2 \),
\[
|I| = \left| (|\eta - \nabla u_0|^{p(x)} - 2(\eta - \nabla u_0) - |\eta|^{p(x) - 2}\eta) |\eta| \right| \leq c_0 |\nabla u_0|^{p(x) - 1} |\eta| \\
\leq C|\eta| \leq \frac{1}{2} |\eta|^{p(x)} + C,
\]
where \( C \) is a generic positive constant independent of \( \varepsilon \).

Thus we have
\[
A_\varepsilon(x, \eta)\eta \geq |\eta|^{p(x)} - \frac{\lambda_\varepsilon}{1 - \lambda_\varepsilon} |I| \geq \frac{1}{2} |\eta|^{p(x)} - C.
\]

It follows from Theorem 4.1 in [6] that \( u_\varepsilon \in L^\infty(\Omega) \) and \( |u_\varepsilon|_{L^\infty(\Omega)} \leq C \), because \( \|u_\varepsilon\|_{W_0^{1, p(x)}(\Omega)} \) is bounded uniformly for \( \varepsilon \in (0, 1) \), where \( C \) is a positive constant independent of \( \varepsilon \). Below we shall prove that \( \|u_\varepsilon\|_{C^0_{\alpha}(\Omega)} \leq C \) for some \( \alpha \in (0, 1) \) by using Theorem 1.2 in [4] in the following two cases, respectively.

**Case (I):** \( \lambda_\varepsilon \in [-1, 0] \).

Noting that \( u_0 \) satisfies the equation, \( -\text{div}(|\nabla u_0|^{p(x) - 2}\nabla u_0) = v_0 \), where \( v_0 \in L^{\beta(\xi)}(\Omega) \) and \( v_0(x) \in \partial F(x, u_0(x)) \) for almost all \( x \in \Omega \). Equation (3.4) is equivalent to the following equation:
\[
-\text{div}\{|\nabla u_\varepsilon|^{p(x) - 2}\nabla u_\varepsilon - \lambda_\varepsilon|\nabla u_\varepsilon - \nabla u_0|^{p(x) - 2}(\nabla u_\varepsilon - \nabla u_0) \} = w_\varepsilon - \lambda_\varepsilon v_0.
\]

Define \( \overline{B}_\varepsilon(x, \eta) = w_\varepsilon - \lambda_\varepsilon v_0 \),
\[
\overline{A}_\varepsilon(x, \eta) = |\eta|^{p(x) - 2}\eta - \lambda_\varepsilon|\eta - \nabla u_0|^{p(x) - 2}(\eta - \nabla u_0) - \lambda_\varepsilon|\nabla u_0|^{p(x) - 2}\nabla u_0.
\]

Then \( u_\varepsilon \) is a solution of the following problem:
\[
\begin{cases}
-\text{div}\overline{A}_\varepsilon(x, \nabla u) = \overline{B}_\varepsilon(x, u), & \text{a.a. } x \in \Omega, \\
|u|_{\partial \Omega} = 0.
\end{cases}
\]

We can prove that, for \( x, y \in \overline{\Omega}, \eta \in \mathbb{R}^N \setminus \{0\}, \xi \in \mathbb{R}^N, t \in \mathbb{R} \), the following estimations hold:
\[
\begin{cases}
\overline{A}_\varepsilon(x, 0) = 0, \\
\frac{\partial (\overline{A}_\varepsilon)}{\partial \eta_i}(x, \eta)\xi_i \xi_j \geq c_7 |\eta|^{p(x) - 2} |\xi|^2, \\
\sum_{i,j=1}^n \frac{\partial (\overline{A}_\varepsilon)}{\partial \eta_i}(x, \eta)\xi_j |\eta| \leq c_8 (1 + |\eta|^{p(x) - 1}), \\
|\overline{B}_\varepsilon(x, \eta)| \leq c_9 + c_{10}|t|^{\alpha(x) - 2} \leq c_{11} + c_{12}|t|^{\alpha(x)},
\end{cases}
\]
and for sufficiently small \( \delta > 0 \), there exists a positive constant \( c_\delta \), depending on \( p^+, p^- \) and \( \delta \), but independent of \( \lambda_\varepsilon \in [-1, 0] \), such that
\[
|\overline{A}_\varepsilon(x, \eta) - \overline{A}_\varepsilon(y, \eta)| \leq c_\delta |x - y|^\beta (1 + |\eta|^{p_0 - 1 + \delta}),
\]
where \( p_0 = \max \{ p(x), p(y) \} \).

The proof of (3.9) is immediate (see [1]), here we only prove (3.10). It follows from \( p \in C^{0,\beta}(\Omega) \) that

\[
|\eta|^{p(x)-2}\eta - |\eta|^{p(y)-2}\eta \leq \left| |\eta|^{p(x)-1} - |\eta|^{p(y)-1} \right| \\
\leq c|x - y|^{\beta}(1 + |\eta|^{p_0-1})(1 + |\log|\eta||).
\]

For given \( \delta \in (0, p^- - 1) \), by \( \lim_{t \to +\infty} \frac{\log \delta}{t} = 0 \) and \( \lim_{t \to 0^+} \frac{\log \delta}{t} = 0 \), we have that there exists a positive constant \( c(\delta) \), depending only on \( \delta \), such that

\[
|\log|\eta|| \leq c(\delta) + |\eta|^{\delta} + |\eta|^{-\delta}
\]

for every \( \eta \in \mathbb{R}^N \). Thus we obtain

\[
|\eta|^{p(x)-2}\eta - |\eta|^{p(y)-2}\eta \leq c_\delta|x - y|^{\beta}(1 + |\eta|^{p_0-1+\delta}).
\]

Similarly, we have

\[
|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - |\eta - \nabla u_0|^{p(y)-2}(\eta - \nabla u_0)| \\
\leq c_\delta|x - y|^{\beta}(1 + |\eta - \nabla u_0|^{p_0-1+\delta}) \\
\leq c_\delta|x - y|^{\beta}(1 + |\eta|^{p_0-1+\delta})
\]

and

\[
|\nabla u_0|^{p(x)-2}(\nabla u_0) - |\nabla u_0|^{p(y)-2}(\nabla u_0)| \leq c_\delta|x - y|^{\beta}(1 + |\nabla u_0|^{p_0-1+\delta}) \\
\leq c_\delta|x - y|^{\beta}.
\]

Hence, noting that \( |\lambda_\varepsilon| \leq 1 \), we see that (3.10) is true.

**Case (II):** \( \lambda_\varepsilon < -1 \).

From (3.4), we have

\[
-\text{div} \left( -\frac{1}{\lambda_\varepsilon} |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon + |\nabla u_\varepsilon - \nabla u_0|^{p(x)-2}(\nabla u_\varepsilon - \nabla u_0) \right) = -\frac{1}{\lambda_\varepsilon} w_\varepsilon.
\]

Note that

\[
-\text{div} \left( \frac{1}{\lambda_\varepsilon} |\nabla u_0|^{p(x)-2}\nabla u_0 \right) = \frac{1}{\lambda_\varepsilon} v_0.
\]

So,

\[
-\text{div} \left[ |\nabla u_\varepsilon - \nabla u_0|^{p(x)-2}(\nabla u_\varepsilon - \nabla u_0) - \frac{1}{\lambda_\varepsilon} |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \\
+ \frac{1}{\lambda_\varepsilon} |\nabla u_0|^{p(x)-2}\nabla u_0 \right] = -\frac{1}{\lambda_\varepsilon} (w_\varepsilon - v_0).
\]

Then \( u_\varepsilon \) is a solution of the following problem:

\[
\begin{aligned}
-\text{div}A_\varepsilon(x, \nabla u) &= \widetilde{B}_\varepsilon(x, u), \quad \text{a.a. } x \in \Omega, \\
|u|_{\partial \Omega} &= 0,
\end{aligned}
\]

where \( \widetilde{B}_\varepsilon(x, t) = -\frac{1}{\lambda_\varepsilon}(w_\varepsilon - v_0) \),
\[
\widetilde{A}_\varepsilon(x, \eta) = |\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - \frac{1}{\lambda_\varepsilon}|\eta|^{p(x)-2}\eta + \frac{1}{\lambda_\varepsilon}|\nabla u_0|^{p(x)-2}\nabla u_0.
\]

Analogously to the case (I), we can prove that \( \widetilde{A}_\varepsilon \) and \( \widetilde{B}_\varepsilon \) satisfy the corresponding conditions (3.9).

From the analysis in case (I) and case (II), we know that Theorem 1.2 in [4] is available. Hence \( u_\varepsilon \in C^{1,\alpha}_0(\Omega) \) and \( \|u_\varepsilon\|_{C^{1,\alpha}_0(\Omega)} \leq C \), that is \( u_\varepsilon \rightarrow u_0 \) in \( C^1(\Omega) \) (by \( C^{1,\alpha}_0(\Omega) \hookrightarrow C^1(\Omega) \) compact embedding) as \( \varepsilon \rightarrow 0 \). The proof of Theorem 1 is complete. \( \Box \)

References


