Solvability of Boundary Value Problems for
Singular Quasi-Laplacian Differential Equations
on the Whole Line*

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Received July 6, 2011; revised March 26, 2012; published online June 1, 2012

Abstract. This paper is concerned with some integral type boundary value problems associated to second order singular differential equations with quasi-Laplacian on the whole line. The emphasis is put on the one-dimensional $p$-Laplacian term $[\Phi(\rho(t))a(t,x(t),x'(t))x'(t)]'$ involving a nonnegative function $\rho$ that may be singular at $t = 0$ and such that $\int_{-\infty}^{0} \frac{ds}{\rho(s)} = \int_{0}^{+\infty} \frac{ds}{\rho(s)} = +\infty$. A Banach space and a nonlinear completely continuous operator are defined in this paper. By using the Schauder’s fixed point theorem, sufficient conditions to guarantee the existence of at least one solution are established. An example is presented to illustrate the main theorem.

Keywords: second order singular differential equation with quasi-Laplacian on the whole line, integral type boundary value problem, fixed point theorem.

AMS Subject Classification: 34B10; 34B15; 35B10.

1 Introduction

The multi-point boundary-value problems for linear second order ordinary differential equations (ODEs) were initiated by Il’in and Moiseev [15]. Since then, more general nonlinear multi-point boundary-value problems (BVPs) were studied by several authors, see the paper [8, 9, 10, 19], the text books [1, 13, 14], the survey papers [11, 18] and the references therein. However, the study of the existence of solutions of differential equations on the whole real line with nonlinear differential operators does not seem to be sufficiently developed [5].

Differential equations governed by nonlinear differential operators have been widely studied. In this setting the most investigated operator is the classical $p$-Laplacian, that is $\Phi_p(x) = |x|^{p-2}x$ with $p > 1$, which, in recent years, has been

* Supported by the Natural Science Foundation of Guangdong province (No. S2011010001900) and the Guangdong Higher Education Foundation for High-level talents.
generalized to other types of differential operators, that preserve the monotonicity of the $p$-Laplacian, but are not homogeneous. These more general operators, which are usually referred to as $\Phi$-Laplacian (or quasi-Laplacian), are involved in some models, e.g. in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. The related nonlinear differential equation has the form

$$[\Phi(x')]' = f(t, x, x'), \quad t \in (-\infty, +\infty),$$

where $\Phi : R \to R$ is an increasing homeomorphism such that $\Phi(0) = 0$. More recently, equations involving other types of differential operators have been studied from a different point of view arising from other types of models, e.g. reaction diffusion equations with non-constant diffusivity and porous media equations. This leads to consider nonlinear differential operators of the type $[a(t, x, x')\Phi(x')]'$, where $a$ is a positive continuous function. For a comprehensive bibliography on this subject, see e.g. [11, 16, 18].

In [17], the authors study a class of BVPs for the second order nonlinear ordinary differential equations on the whole line. Two theorems have been proved. The first one is established by the use of the Schauder theorem and concerns the existence of solutions, while the second one deals with the existence and uniqueness of solutions and is derived by the Banach contraction principle.

In [12], the authors study the boundary value problem $[a(x(t))\Phi(x'(t))]' = f(t, x(t), x'(t)), \quad t \in (-\infty, +\infty), \quad x(-\infty) = \nu_1, \quad x(+\infty) = \nu_2$, establishing the existence and non-existence of heteroclinic solutions.

In [5], Bianconi and Papalini investigate the existence of solutions of the following boundary value problem

$$\lim_{t \to -\infty} x(t) =: x(-\infty) = 0, \quad \lim_{t \to +\infty} x(t) =: x(+\infty) = 1,$$

where $\Phi$ is a monotone function which generalizes the one-dimensional $p$-Laplacian operator. A criterion for the existence and non-existence of solutions of BVP (1.1) is established. In [2, 4], Avramescu and Vladimirescu study the following boundary value problem

$$x''(t) + 2f(t)x'(t) + x(t) + g(t, x(t)) = 0, \quad t \in R, \quad \lim_{t \to \pm \infty} x(t) =: x(\pm \infty) = 0,$$

where $f$ and $g$ are given functions. The existence of solutions of BVP (1.2) is obtained. In [3], Avramescu and Vladimirescu study the following boundary value problem

$$x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in R, \quad \lim_{t \to \pm \infty} x(t) = \lim_{t \to \pm \infty} x'(t), \quad \lim_{t \to \pm \infty} x'(t) = \lim_{t \to \pm \infty} x'(t),$$

under some adequate hypotheses and using the Bohnenblust–Karlin fixed point theorem, the existence of solutions of BVP (1.3) is established.
Solvability of BVPs for Singular Quasi-Laplacian Differential Equations

Cabada and Cid [6] prove the solvability of the boundary value problem on the whole line
\[
\left[\Phi(x'(t))\right]' + f(t, x(t), x'(t)) = 0, \quad t \in R, \\
\lim_{t \to -\infty} x(t) = -1, \quad \lim_{t \to +\infty} x(t) = 1,
\]
where \(f\) is a continuous function, \(\Phi: (-a, a) \to R\) is a homeomorphism with \(a \in (0, +\infty)\), i.e., \(\Phi\) is singular. Calamai [7] and Marcelli, Papalini [17] discuss the solvability of the following strongly nonlinear BVP:
\[
\left[a(x(t))\Phi(x'(t))\right]' + f(t, x(t), x'(t)) = 0, \quad t \in R, \\
\lim_{t \to -\infty} x(t) = \alpha, \quad \lim_{t \to +\infty} x(t) = \beta,
\]
where \(\alpha < \beta\), \(\Phi\) is a general increasing homeomorphism with bounded domain (singular \(\Phi\)-Laplacian), \(a\) is a positive continuous function and \(f\) is a Carathéodory nonlinear function. Conditions for the existence and nonexistence of heteroclinic solutions in terms of the behavior of \(y\) imply that \(y \to \Phi(y)\) as \(y \to 0\), and of \(t \to f(t, x, y)\) as \(|t| \to +\infty\) are established. The approach is based on fixed point techniques suitably combined to the method of upper and lower solutions.

Motivated by the mentioned papers, we consider the more general BVP for a second order singular differential equation on the whole line with quasi-Laplacian operator
\[
\left[\Phi(\rho(t)a(t, x(t), x'(t))x'(t))\right]' + f(t, x(t), x'(t)) = 0, \quad t \in R, \\
\lim_{t \to -\infty} \rho(t)a(t, x(t), x'(t))x'(t) - \int_{-\infty}^{+\infty} \alpha(s)x(s) ds = \int_{-\infty}^{+\infty} g(s, x(s), x'(s)) ds, \\
\lim_{t \to +\infty} \rho(t)a(t, x(t), x'(t))x'(t) + \int_{-\infty}^{+\infty} \beta(s)x(s) ds = \int_{-\infty}^{+\infty} h(s, x(s), x'(s)) ds,
\]
where
- \(\rho \in C^0(R, [0, +\infty))\) with \(\rho(t) > 0\) for all \(t \neq 0\) satisfies
  \[
  \int_{-\infty}^{0} ds/\rho(s) = +\infty, \quad \int_{0}^{+\infty} ds/\rho(s) = +\infty.
  \]
Denote \(\tau(t) = \left|\int_{0}^{t} ds/\rho(s)\right|\).
- \(a : R \times R \times R \to (0, +\infty)\) is continuous and satisfies that there exist constants \(m > 0, M > 0\) such that
  \[
  m \leq a(t, (1 + \tau(t))x, y/\rho(t)) \leq M, \quad t \in R, \; x \in R, \; y \in R
  \]
  and for each \(r > 0, |x|, |y| \leq r\) imply that \(a(t, (1 + \tau(t))x, y/\rho(t)) \to a_{\pm\infty}\) uniformly as \(t \to \pm\infty\).
- \(f, g, h\) defined on \(R^3\) are nonnegative Carathéodory functions.

\( \alpha, \beta : R \to [0, +\infty) \) are continuous functions satisfying
\[
\int_{-\infty}^{+\infty} \alpha(s) \, ds > 0, \quad \int_{-\infty}^{+\infty} \frac{\beta(s)}{\rho(s)} \, ds < +\infty,
\]
\[
\int_{0}^{+\infty} \alpha(s) \int_{0}^{s} \frac{dr}{\rho(r)} \, ds < +\infty, \quad \int_{-\infty}^{0} \alpha(s) \int_{s}^{0} \frac{dr}{\rho(r)} \, ds < +\infty.
\]

\( \Phi \in C^1(R) \) (a quasi-Laplacian operator) is continuous and strictly increasing on \( R \), \( \Phi(0) = 0 \) and its inverse function denoted by \( \Phi^{-1} \) is continuous too, moreover \( \Phi^{-1} \) satisfies that there exist constants \( L > 0 \) and \( L_n > 0 \) such that
\[
\Phi^{-1}(x_1 + \cdots + x_n) \leq L\Phi^{-1}(x_1) + \cdots + \Phi^{-1}(x_n), \quad x_i \geq 0 \ (i = 1, 2, \ldots, n).
\]

It is well known that \( \Phi(s) = |s|^{p-2}s \) with \( p > 1 \) is called \( p \)-Laplacian. One sees that quasi-Laplacian contains \( p \)-Laplacian as special case. But \( \Phi(s) = \frac{s^3}{1+s^2} \) is a quasi-Laplacian not a \( p \)-Laplacian.

By a solution of BVP (1.7) we mean a function \( x \in C^1(R) \) such that
\[
\Phi(\rho ax') : t \to \Phi(\rho(t)a(t, x(t), x'(t)))x'(t)
\]
belongs to \( W^{1,1}(R) \) and all equations in (1.7) are satisfied.

The purpose is to establish sufficient conditions for the existence of at least one solution of BVP (1.7). The results in this paper generalize and improve some known ones since the quasi-Laplacian term \( [\Phi(\rho(t)a(t, x(t), x'(t)))x'(t)]' \) involves the nonnegative function \( \rho \) that may satisfy \( \rho(0) = 0 \).

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3. An example is presented in Section 4 to illustrate the prototype of the main theorem.
Lemma 2. Suppose that $x \in X$. Denote

$$
\sigma_1 = -\int_{-\infty}^{+\infty} |f(r, x(r), x'(r))| \, dr + \Phi\left( \frac{\int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr}{1 + \int_{-\infty}^{+\infty} \frac{\beta(r)}{\rho(r) a(r, x(r), x'(r))} \, dr} \right),
$$

$$
\sigma_2 = \int_{-\infty}^{+\infty} |f(r, x(r), x'(r))| \, dr + \Phi\left( \frac{\int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr}{1 + \int_{-\infty}^{+\infty} \frac{\beta(r)}{\rho(r) a(r, x(r), x'(r))} \, dr} \right).
$$

Then there exists a unique constant $A_x \in [\sigma_1, \sigma_2]$ such that

$$
\Phi^{-1}(A_x) + \int_{-\infty}^{+\infty} \frac{\beta(s) \Phi^{-1}(A_x) + \int_{s}^{+\infty} f(r, x(r), x'(r)) \, dr}{\rho(s) a(s, x(s), x'(s))} \, ds
$$

$$
- \int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr = 0.
$$

(2.1)

Furthermore, it holds that

$$
|A_x| \leq \int_{-\infty}^{+\infty} |f(r, x(r), x'(r))| \, dr + \Phi\left( \frac{\int_{-\infty}^{+\infty} |h(r, x(r), x'(r))| \, dr}{1 + \int_{-\infty}^{+\infty} \frac{\beta(r)}{M \rho(r)} \, dr} \right),
$$

(2.2)

where $M$ is defined in Section 1.

Proof. Since $x \in X$, $f$, $h$ are Caratheodory functions, then

$$
\|x\| = \max\left\{ \sup_{t \in R} \frac{|x(t)|}{1 + \tau(t)}, \sup_{t \in R} \rho(t)|x'(t)| \right\} = r < +\infty,
$$

and both

$$
\int_{-\infty}^{+\infty} f(r, x(r), x'(r)) \, dr \quad \text{and} \quad \int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr
$$
converge. Let
\[ G(c) = \Phi^{-1}(c) + \int_{-\infty}^{+\infty} \frac{\beta(s)\Phi^{-1}(c + \int_{s}^{+\infty} f(r, x(r), x'(r)) \, dr)}{\rho(s) a(s, x(s), x'(s))} \, ds \]
\[ - \int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr. \]
Since \( \int_{-\infty}^{+\infty} \frac{\beta(s)}{\rho(s)} \, ds < +\infty \), then \( G(c) \) is well defined on \( R \). It is easy to see that \( G(c) \) is strictly increasing on \( R \). We find that
\[ G(\sigma_1) = \Phi^{-1} \left( -\int_{-\infty}^{+\infty} |f(r, x(r), x'(r))| \, dr + \Phi \left( \frac{\int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr}{1 + \int_{-\infty}^{+\infty} \frac{\beta(r)}{\rho(r) a(r, x(r), x'(r))} \, dr} \right) \right) \]
\[ + \int_{-\infty}^{+\infty} \frac{\beta(s)}{\rho(s) a(s, x(s), x'(s))} \, ds \Phi^{-1} \left( \Phi \left( \frac{\int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr}{1 + \int_{-\infty}^{+\infty} \frac{\beta(r)}{\rho(r) a(r, x(r), x'(r))} \, dr} \right) \right) \]
\[ - \int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr \]
\[ = \frac{\int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr}{1 + \int_{-\infty}^{+\infty} \frac{\beta(r)}{\rho(r) a(r, x(r), x'(r))} \, dr} - \int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr \]
\[ + \int_{-\infty}^{+\infty} \frac{\beta(s)}{\rho(s) a(s, x(s), x'(s))} \, ds \int_{-\infty}^{+\infty} \frac{h(r, x(r), x'(r)) \, dr}{1 + \int_{-\infty}^{+\infty} \frac{\beta(r)}{\rho(r) a(r, x(r), x'(r))} \, dr} = 0. \]

Similarly we find that \( G(\sigma_2) \geq 0 \).

Hence there exists a unique constant \( A_x \in [\sigma_1, \sigma_2] \) such that (2.1) holds. It is easy to see from \( A_x \in [\sigma_1, \sigma_2] \) that (2.2) holds. The proof is complete. \( \square \)

Define the operator \( T : X \to X \) by
\[
(Tx)(t) = \begin{cases} 
B_x + \int_0^t \frac{\Phi^{-1}(A_x + \int_s^{+\infty} f(r, x(r), x'(r)) \, dr)}{\rho(s) a(s, x(s), x'(s))} \, ds, & t \geq 0, \\
B_x - \int_t^0 \frac{\Phi^{-1}(A_x + \int_s^{+\infty} f(r, x(r), x'(r)) \, dr)}{\rho(s) a(s, x(s), x'(s))} \, ds, & t \leq 0, 
\end{cases} \tag{2.3}
\]
where \( A_x \) satisfies (2.1) and
\[ B_x = \frac{\Phi^{-1}(A_x + \int_{-\infty}^{+\infty} f(r, x(r), x'(r)) \, dr) - \int_{-\infty}^{+\infty} g(r, x(r), x'(r)) \, dr}{\int_{-\infty}^{+\infty} \alpha(s) \, ds}. \]
Lemma 3. The following properties hold:

(i) $T x$ satisfies

$$\lim_{t \to -\infty} \rho(t) a(t, x(t), x'(t))(T x)'(t) = f(t, x(t), x'(t)) = 0, \quad t \in R,$$

and

$$\int_{-\infty}^{+\infty} f(r, x(r), x'(r)) \, dr, \quad \int_{-\infty}^{+\infty} g(r, x(r), x'(r)) \, dr, \quad \int_{-\infty}^{+\infty} h(r, x(r), x'(r)) \, dr$$

converge. From the definitions of $A_x$ and $B_x$, we get

$$\rho(t)(T x)'(t) = \frac{1}{a(t, x(t), x'(t))} \Phi^{-1}\left( A_x + \int_t^{+\infty} f(r, x(r), x'(r)) \, dr \right).$$

It is easy to see that (2.5) holds.

(ii) From the assumptions imposed on $\alpha, \beta, \rho$, we know that $t \to (T x)(t)$ is continuous on $R$ and $(T x)(t)/(1 + \tau(t))$ is bounded on $R$. Furthermore,

$$\rho(t)(T x)'(t) = \frac{\Phi^{-1}(A_x + \int_t^{+\infty} f(r, x(r), x'(r)) \, dr)}{a(t, x(t), x'(t))}.$$ 

It is easy to see that $t \to \rho(t)(T x)'(t)$ is continuous on $R$ and $\rho(t)(T x)'(t)$ is bounded on $R$. It follows that $T x \in X$. Hence $T : X \to X$ is well defined.
(iii) It is easy to see that \( x \in X \) is a solution of BVP (1.7) if and only if \( x \) is a fixed point of \( T \) in \( X \).

(iv) The following five steps are needed (Steps 1–2 imply that \( T : X \to X \) is continuous and Steps 3–5 imply that \( T \) maps bounded sets into relatively compact sets). It follows that \( T : X \to X \) is completely continuous.

**Step 1.** We prove that the function \( A_x : X \to R \) is continuous in \( x \).

Let \( \{x_n\} \in X \) with \( x_n \to x_0 \) as \( n \to \infty \). Let \( \{A_{x_n}\} (n = 0, 1, 2, \ldots) \) be constants decided by equation

\[
\Phi^{-1}(A_{x_n}) + \int_{-\infty}^{+\infty} \frac{\beta(s)\Phi^{-1}(A_{x_n} \Phi(s) + \int_{s}^{+\infty} f(r, x_n(r), x_n'(r)) dr)}{\rho(s)a(s, x_n(s), x_n'(s))} ds
- \int_{-\infty}^{+\infty} h(r, x_n(r), x_n'(t)) dr = 0.
\]

Corresponding to \( x_n \) \((n = 0, 1, 2, \ldots)\). Since \( x_n \to x_0 \) as \( n \to \infty \), there exists an \( M_0 > 0 \) such that \( \|x_n\| \leq M_0 \) \((n = 0, 1, 2, \ldots)\). The fact \( f, g, h \) are Carathéodory functions means there exists \( \phi_{M_0} \in L^1(R) \) such that

\[
f(t, x_n(t), x_n'(t)) = f(t, x(t), \frac{1}{\rho(t)} \phi(t)x_n'(t)) \leq \phi_{M_0}(t), \quad t \in R,
\]

\[
g(t, x(t), x'(t)) \leq \phi_{M_0}(t), \quad h(t, x(t), x'(t)) \leq \phi_{M_0}(t), \quad t \in R.
\]

Then

\[
\int_{-\infty}^{+\infty} f(r, x_n(r), x_n'(r)) dr \leq \int_{-\infty}^{+\infty} \phi_{M_0}(r) dr < \infty,
\]

\[
\int_{-\infty}^{+\infty} g(r, x_n(r), x_n'(r)) dr \leq \int_{-\infty}^{+\infty} \phi_{M_0}(r) dr < \infty,
\]

\[
\int_{-\infty}^{+\infty} h(r, x_n(r), x_n'(r)) dr \leq \int_{-\infty}^{+\infty} \phi_{M_0}(r) dr < \infty.
\]

So, by (2.2), we have

\[
|A_{x_n}| \leq \int_{-\infty}^{+\infty} |f(r, x(r), x'(r))| ds + \Phi\left(\int_{-\infty}^{+\infty} |h(r, x(r), x'(r))| dr\right) \frac{1 + \int_{-\infty}^{+\infty} \beta(r) dr}{\int_{-\infty}^{+\infty} M \rho(r) dr}
\]

\[
\leq \int_{-\infty}^{+\infty} \phi_{M_0}(s) ds + \Phi\left(\int_{-\infty}^{+\infty} \phi_{M_0}(s) ds \frac{1 + \int_{-\infty}^{+\infty} \beta(r) dr}{\int_{-\infty}^{+\infty} M \rho(r) dr} \right),
\]

which means that \( \{A_{x_n}\} \) is uniformly bounded. It follows that

\[
\int_{-\infty}^{+\infty} \frac{\beta(s)\Phi^{-1}(A_{x_n} \Phi(s) + \int_{s}^{+\infty} f(r, x_n(r), x_n'(r)) dr)}{\rho(s)a(s, x_n(s), x_n'(s))} ds
\]

\[
\leq \frac{1}{m} \int_{-\infty}^{+\infty} \beta(s) ds \Phi^{-1}\left(2 \int_{-\infty}^{+\infty} \phi_{M_0}(r) dr + \Phi\left(\int_{-\infty}^{+\infty} \phi_{M_0}(s) ds \frac{1 + \int_{-\infty}^{+\infty} \beta(r) dr}{\int_{-\infty}^{+\infty} M \rho(r) ds} \right)\right).
\]
Suppose that \( \{ A_{x_n} \} \) does not converge to \( A_{x_0} \). Then there exist two subsequences \( \{ A_{x_{n_k}}^{(1)} \} \) and \( \{ A_{x_{n_k}}^{(2)} \} \) of \( \{ A_{x_n} \} \) with \( A_{x_{n_k}}^{(1)} \to c_1 \) and \( A_{x_{n_k}}^{(2)} \to c_2 \) as \( k \to \infty \), but \( c_1 \neq c_2 \). By the construction of \( A_{x_n} \) \( (n = 1, 2, \ldots) \), we have

\[
\phi^{-1}(A_{x_{n_k}}^{(1)}) + \int_{-\infty}^{+\infty} \frac{\beta(s)\phi^{-1}(A_{x_{n_k}}^{(1)}) + \int_{s}^{+\infty} f(r, x_{n_k}^{(1)}(r), x_{n_k}^{(1)'}(r)) \, dr}{\rho(s)\alpha(s, x_{n_k}(s), x_{n_k}^{(1)'}(s))} \, ds
- \int_{-\infty}^{+\infty} h(r, x_{n_k}^{(1)}(r), x_{n_k}^{(1)'}(t)) \, dr = 0.
\]

Let \( k \to \infty \), using Lebesgue’s dominated convergence theorem, the above equality implies

\[
\phi^{-1}(A_{x_0}) + \int_{-\infty}^{+\infty} \frac{\beta(s)\phi^{-1}(A_{x_0}) + \int_{s}^{+\infty} f(r, x_0(r), x_0'(r)) \, dr}{\rho(s)\alpha(s, x_0(s), x_0'(s))} \, ds
- \int_{-\infty}^{+\infty} h(r, x_0(r), x_0'(t)) \, dr = 0.
\]

Since \( \{ A_{x_0} \} \) is unique with respect to \( x_0 \), we get \( c_1 = A_{x_0} \). Similarly, \( c_2 = A_{x_0} \). Thus \( c_1 = c_2 \), a contradiction. So, for any \( x_n \to x_0 \), one has \( A_{x_n} \to A_{x_0} \), which means \( A_{x_0} : X \to R \) is continuous.

**Step 2.** We show that \( T \) is continuous on \( X \). Since \( A_x \) is continuous, then \( B_x \) is continuous too. From the continuity of \( A_x \) and \( B_x \), and since \( f, g, h \) are Carathéodory functions, the result follows.

To prove that \( T \) maps bounded sets into relatively compact sets, we must prove that \( TD \) is relatively compact. Recall \( W \subset X \) is relatively compact if

(i) it is bounded,

(ii) both \( \{ \frac{T_x}{1+\tau(t)} : x \in W \} \) and \( \{ \rho(t)(T_x)' : x \in W \} \) are equi-continuous on any closed subinterval of \( (-\infty, +\infty) \),

(iii) both \( \{ \frac{T_x}{1+\tau(t)} : x \in W \} \) and \( \{ \rho(t)(T_x)' : x \in W \} \) are equi-convergent at \( t = -\infty \),

(iv) both \( \{ \frac{T_x}{1+\tau(t)} : x \in W \} \) and \( \{ \rho(t)(T_x)' : x \in W \} \) are equi-convergent at \( t = +\infty \).

Hence we must do the following three steps.

**Step 3.** We show that \( T \) maps bounded subsets into bounded sets. Let \( D \subset X \) be a given bounded set. Then, there exists \( M_0 > 0 \) such that \( D \subset \{ x \in X : \|x\| \leq M_0 \} \). Then there exists \( \phi_{M_0} \in L^1(R) \) such that

\[
|f(t, x(t), x'(t))| = |f(t, x(t), \frac{1}{\rho(t)}\rho(t)x'(t))| \leq \phi_{M_0}(t), \quad t \in R,
\]

\[
g(t, x(t), x'(t)) \leq \phi_{M_0}(t), \quad |h(t, x(t), x'(t))| \leq \phi_{M_0}(t), \quad t \in R.
\]

Therefore,

\[
\int_{-\infty}^{+\infty} |f(r, x(r), x'(r))| \, dr \leq \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr < \infty,
\]

\[
\int_{-\infty}^{+\infty} |g(r, x(r), x'(r))| \, dr \leq \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr < \infty,
\]

\[
\int_{-\infty}^{+\infty} |h(r, x(r), x'(r))| \, dr \leq \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr < \infty.
\]

Similarly we have

\[
|A_x| \leq \int_{-\infty}^{+\infty} \phi_{M_0}(s) \, ds + \Phi \left( \int_{-\infty}^{+\infty} \phi_{M_0}(s) \, ds \right),
\]

\[
|B_x| = \left| \Phi^{-1}(A_x + \int_{-\infty}^{+\infty} f(r, x(r), x'(r)) \, dr) - \int_{-\infty}^{+\infty} g(r, x(r), x'(r)) \, dr \right|
\]

\[
\leq \phi_{M_1} + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr
\]

\[
\leq \Phi^{-1}(M_1 + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr) + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr
\]

\[
\leq \Phi^{-1}(M_1 + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr)
\]

\[
\leq M_2 < +\infty.
\]

Therefore,

\[
\frac{|(T\alpha)(t)|}{1 + \tau(t)} = \begin{cases} 
\int_{0}^{t} \frac{\phi^{-1}(A_x + \int_{-\infty}^{+\infty} f(r, x(r), x'(r)) \, dr)}{\rho(s)a(u, x(u), x'(u))} \, ds, & t \geq 0, \\
\int_{t}^{0} \frac{\phi^{-1}(A_x + \int_{-\infty}^{+\infty} f(r, x(r), x'(r)) \, dr)}{\rho(s)a(u, x(u), x'(u))} \, ds, & t \leq 0,
\end{cases}
\]

\[
\leq \begin{cases} 
M_2 + \frac{1}{m} \int_{0}^{t} \frac{1}{\rho(s)} \, ds \phi^{-1}(M_1 + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr), & t \geq 0, \\
M_2 + \frac{1}{m} \int_{t}^{0} \frac{1}{\rho(s)} \, ds \phi^{-1}(M_1 + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr), & t \leq 0,
\end{cases}
\]

\[
\leq M_2 + \frac{1}{m} \phi^{-1}(M_1 + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr), \quad t \geq 0,
\]

\[
M_2 + \frac{1}{m} \phi^{-1}(M_1 + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr), \quad t \leq 0,
\]

\[
=: M_3 < +\infty.
\]
On the other hand, we have
\[
\rho(t) |(T x)'(t)| = \frac{|\Phi^{-1}(A_x + \int_{t_1}^{+\infty} f(r, x(r), x'(r)) dr)|}{a(t_1, x(t_1), x'(t_1))} \leq \frac{1}{m} \Phi^{-1} \left( M_1 + \int_{-\infty}^{+\infty} \phi M_0(r) dr \right) =: M_4.
\]

Then
\[
\|T x\| = \max \left\{ \sup_{t \in R} \frac{|(T x)(t)|}{1 + \tau(t)}, \sup_{t \in R} \rho(t) |(T x)'(t)| \right\} \leq \max \{M_3, M_4\} < +\infty.
\]

So, \(\{T x: x \in D\}\) is bounded.

**Step 4.** Let \(D\) be a bounded subset of \(X\). We prove that both \(\{\frac{T x}{1+\tau(t)}: x \in D\}\) and \(\{\rho(T x)': x \in D\}\) are equi-continuous on each finite subinterval \([-K, K]\) on \(R\). Suppose that \(D \subset \{x \in X: \|x\| \leq M_0\}\). For any \(K > 0, t_1, t_2 \in [-K, K]\) with \(t_1 < t_2\) and \(x \in X\), since \(f, g, h\) are Caratheodory functions, then there exists \(\phi M_0 \in L^1(R)\) such that (2.7) and (2.8) hold. One sees that (2.6) holds.

First, we consider \(\|\rho(t_1)(T x)'(t_1) - \rho(t_2)(T x)'(t_2)\|\). One sees that
\[
\|\rho(t_1)(T x)'(t_1) - \rho(t_2)(T x)'(t_2)\|
\leq \left| \Phi^{-1}(A_x + \int_{t_1}^{+\infty} f(r, x(r), x'(r)) dr) \right| \left| \frac{1}{a(t_1, x(t_1), x'(t_1))} - \frac{1}{a(t_2, x(t_2), x'(t_2))} \right|
\leq \left| \Phi^{-1}(A_x + \int_{t_1}^{+\infty} f(r, x(r), x'(r)) dr) \right| \left| \frac{1}{a(t_1, x(t_1), x'(t_1))} - \frac{1}{a(t_2, x(t_2), x'(t_2))} \right|
\leq \left| \Phi^{-1}(A_x + \int_{t_1}^{+\infty} f(r, x(r), x'(r)) dr) \right| \left| \frac{1}{a(t_1, x(t_1), x'(t_1))} - \frac{1}{a(t_2, x(t_2), x'(t_2))} \right|
\leq \left| \Phi^{-1}(A_x + \int_{t_1}^{+\infty} f(r, x(r), x'(r)) dr) \right| \left| \frac{1}{a(t_1, x(t_1), x'(t_1))} - \frac{1}{a(t_2, x(t_2), x'(t_2))} \right|
\]

Since
\[
\left| A_x + \int_{t}^{+\infty} f(r, x(r), x'(r)) dr \right| \leq \int_{t}^{+\infty} \phi M_0(r) dr + M_1 =: r,
\]
and \(\Phi^{-1}(s)\) is uniformly continuous on \([-r, r]\), then for each \(\epsilon > 0\) there exists \(\mu > 0\) such that \(|s_1 - s_2| < \mu\) with \(s_1, s_2 \in [-r, r]\) implies that \(|\Phi^{-1}(s_1) - \Phi^{-1}(s_2)| < m/2\epsilon\). Since
\[
\left| \Phi(\rho(t_1)a(t_1, x(t_1), x'(t_1))(T x)'(t_1)) - \Phi(\rho(t_2)a(t_2, x(t_2), x'(t_2))(T x)'(t_2)) \right|
\leq \int_{t_1}^{t_2} f(r, x(r), x'(r)) dr \leq \int_{t_1}^{t_2} \phi M_0(r) dr \rightarrow 0 \quad \text{uniformly as} \ t_1 \rightarrow t_2,
\]

then there exists $\sigma_1 > 0$ such that $|t_2 - t_1| < \sigma_1$ implies that

\[
|\Phi(\rho(t_1)a(t_1, x(t_1), x'(t_1))(Tx)'(t_1)) - \Phi(\rho(t_2)a(t_2, x(t_2), x'(t_2))(Tx)'(t_2))| < \mu.
\]

Thus $|t_1 - t_2| < \sigma_1$ implies that

\[
|\rho(t_1)a(t_1, x(t_1), x'(t_1))(Tx)'(t_1) - \rho(t_2)a(t_2, x(t_2), x'(t_2))(Tx)'(t_2)|
\]

\[
= \left|\Phi^{-1}(\Phi(\rho(t_1)a(t_1, x(t_1), x'(t_1))(Tx)'(t_1)))
- \Phi^{-1}(\Phi(\rho(t_2)a(t_2, x(t_2), x'(t_2))(Tx)'(t_2)))\right|
\]

\[
= \left|\Phi^{-1}\left(A_x + \int_{t_1}^{+\infty} f(r, x(r), x'(r)) \, dr\right)
- \Phi^{-1}\left(A_x + \int_{t_2}^{+\infty} f(r, x(r), x'(r)) \, dr\right)\right| < \frac{m}{2\epsilon}.
\]

Since $1/a(t, x(t), x'(t))$ is uniformly continuous on $[-K, K]$, then there exists $\sigma_2 > 0$ such that $|t_2 - t_1| < \sigma_2$ implies

\[
\left|\frac{1}{a(t_1, x(t_1), x'(t_1))} - \frac{1}{a(t_2, x(t_2), x'(t_2))}\right| < \frac{1}{\Phi^{-1}(M_1 + \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr)^2} \epsilon.
\]

Hence $|t_1 - t_2| < \min\{\sigma_1, \sigma_2\}$ with $t_1, t_2 \in [-K, K]$ implies that

\[
|\rho(t_1)(Tx)'(t_1) - \rho(t_2)(Tx)'(t_2)| < \epsilon. \tag{2.9}
\]

Now, we consider $(|Tx(t_1)/(1 + \tau(t_1)) - (Tx(t_2)/(1 + \tau(t_2)))$.

**Case 1.** $0 \leq t_1 \leq t_2 \leq K$. By (2.3), we have

\[
\left|\frac{(Tx)(t_1)}{1 + \tau(t_1)} - \frac{(Tx)(t_2)}{1 + \tau(t_2)}\right| \leq \left|B_x\right| \left|\frac{1}{1 + \tau(t_1)} - \frac{1}{1 + \tau(t_2)}\right|
\]

\[
+ \left|\int_{t_1}^{t_2} \frac{\Phi^{-1}(A_x + \int_{s}^{+\infty} f(r, x(r), x'(r)) \, dr)}{\rho(s)a(s, x(s), x'(s))} \rho(s) \, ds\Phi^{-1}\left(\int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1\right) \right|
\]

\[
\leq M_2 \left|\tau(t_1) - \tau(t_2)\right| + \frac{1}{1 + \tau(t_1)} \frac{1}{m} \int_{t_1}^{t_2} 1 \, ds \Phi^{-1}\left(\int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1\right)
\]

\[
+ \left|\frac{1}{1 + \tau(t_1)} - \frac{1}{1 + \tau(t_2)}\right| \int_{t_1}^{t_2} \frac{1}{\rho(s)} \, ds \Phi^{-1}\left(\int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1\right)
\]

\[
\leq M_2 \left|\tau(t_1) - \tau(t_2)\right| + \frac{1}{m} \int_{t_1}^{t_2} \frac{1}{\rho(s)} \, ds \Phi^{-1}\left(\int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1\right)
\]

\[
+ \frac{1}{m} \left|\frac{1}{1 + \tau(t_1)} - \frac{1}{1 + \tau(t_2)}\right| \int_{t_1}^{t_2} \frac{1}{\rho(s)} \, ds \Phi^{-1}\left(\int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1\right)
\]

\[
\leq M_2 \left|\tau(t_1) - \tau(t_2)\right| + \frac{1}{m} \int_{t_1}^{t_2} \frac{1}{\rho(s)} \, ds \Phi^{-1}\left(\int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1\right)
\]

\[
+ \frac{1}{m} \left|\tau(t_1) - \tau(t_2)\right| \Phi^{-1}\left(\int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1\right).\]
Case 2. \(-K \leq t_1 \leq t_2 \leq 0\). We have similarly that
\[
\left| \frac{(Tx)(t_1)}{1 + \tau(t_1)} - \frac{(Tx)(t_2)}{1 + \tau(t_2)} \right| \\
\leq M_2 |\tau(t_1) - \tau(t_2)| + \frac{1}{m} \int_{t_1}^{t_2} \frac{1}{\rho(s)} \Phi^{-1} \left( \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1 \right) \\
+ \frac{1}{m} |\tau(t_1) - \tau(t_2)| \Phi^{-1} \left( \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1 \right).
\]

Case 3. \(-K \leq t_1 \leq t_2 \leq K\). We have
\[
\left| \frac{(Tx)(t_1)}{1 + \tau(t_1)} - \frac{(Tx)(t_2)}{1 + \tau(t_2)} \right| \\
= |B_x| \left| \frac{1}{1 + \tau(t_1)} - \frac{1}{1 + \tau(t_2)} \right| \\
+ \left| \int_{t_1}^{t_2} \frac{\Phi^{-1}(A_x + \int_{s}^{+\infty} f(r,x(r),x'(r)) \, dr)}{\rho(s)\alpha(s,x(s),x'(s))} \, ds \right| + \left| \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1 \right) \\
+ \frac{1}{m} |\tau(t_1) - \tau(t_2)| \Phi^{-1} \left( \int_{-\infty}^{+\infty} \phi_{M_0}(r) \, dr + M_1 \right).
\]

From Cases 1-3, we get
\[
\left| \frac{(Tx)(t_1)}{1 + \tau(t_1)} - \frac{(Tx)(t_2)}{1 + \tau(t_2)} \right| \to 0 \quad \text{uniformly as } t_1 \to t_2.
\]

Then there exists \(\sigma_3 > 0\) such that \(|t_1 - t_2| < \sigma_3\) with \(t_1, t_2 \in [-K, K]\) implies
\[
\left| \frac{(Tx)(t_1)}{1 + \tau(t_1)} - \frac{(Tx)(t_2)}{1 + \tau(t_2)} \right| < \epsilon. \quad (2.10)
\]

Then (2.9) and (2.10) imply that both \(\{\frac{Tx}{1+\tau(t)} : x \in D\}\) and \(\{\rho(Tx)' : x \in D\}\) are equi-continuous on \([-K, K]\). So both \(\{\frac{Tx}{1+\tau(t)} : x \in D\}\) and \(\{\rho(Tx)' : x \in D\}\) are equi-continuous on each finite subinterval on \(R\).

Step 5. Let \(D\) be a bounded subset of \(X\). We show that both \(\{\frac{Tx}{1+\tau(t)} : x \in D\}\) and \(\{\rho(Tx)' : x \in D\}\) are equi-convergent at \(+\infty\) and \(-\infty\) respectively.
\[
\left| \frac{(Tx)(t)}{1 + \tau(t)} - \frac{\Phi^{-1}(A_x)}{a_+} \right| \\
\leq \frac{|B_x|}{1 + \tau(t)} + \left| \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(A_x + \int_{s}^{+\infty} f(r,x(r),x'(r)) \, dr)}{\rho(s)\alpha(s,x(s),x'(s))} \, ds \right| - \frac{\Phi^{-1}(A_x)}{a_+} \\
\leq M_2 + \frac{\Phi^{-1}(M_1)/a_+}{1 + \tau(t)} + \left| \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(A_x + \int_{s}^{+\infty} f(r,x(r),x'(r)) \, dr)}{\rho(s)\alpha(s,x(s),x'(s))} \, ds - \Phi^{-1}(A_x) \right| a_+ \\
\leq \frac{M_2 + \Phi^{-1}(M_1)/a_+}{1 + \tau(t)} + \left| \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(A_x + \int_{s}^{+\infty} f(r,x(r),x'(r)) \, dr)}{\rho(s)\alpha(s,x(s),x'(s))} \, ds - \Phi^{-1}(A_x) \right| a_+.
\]
It is easy to know that there exists \( T_1 > 0 \) such that \( t > T_1 \) implies

\[
0 < \frac{M_2 + \Phi^{-1}(M_1)/a_+}{1 + \tau(t)} < \frac{\epsilon}{2}, \quad t > T_1.
\]

Similarly to Step 4, we can get that \( \Phi^{-1}(A_x + \int_t^{+\infty} f(r, x(r), x'(r)) \, dr) \) \( \Phi^{-1}(A_x) \) uniformly as \( t \to +\infty \). Together with that

\[
a(t, x(t), x'(t)) = a\left(t, (1 + \tau(t)) \frac{x(t)}{1 + \tau(t)}, \frac{1}{\rho(t)} \rho(t)x'(t)\right) \to a_+
\]

uniformly as \( t \to +\infty \), we know that

\[
\frac{\Phi^{-1}(A_x + \int_t^{+\infty} f(r, x(r), x'(r)) \, dr)}{a(t, x(t), x'(t))} - \frac{\Phi^{-1}(A_x)}{a_+} \to 0 \quad \text{uniformly as } t \to +\infty.
\]

Then there exists \( T_2 > 0 \) such that

\[
\left| \frac{\Phi^{-1}(A_x + \int_t^{+\infty} f(r, x(r), x'(r)) \, dr)}{a(t, x(t), x'(t))} - \frac{\Phi^{-1}(A_x)}{a_+} \right| < \frac{\epsilon}{2}, \quad t > T_2.
\]

Then

\[
\left| \rho(t)(Tx)'(t) - \frac{\Phi^{-1}(A_x)}{a_+} \right| < \frac{\epsilon}{2}, \quad t > T_2. \tag{2.11}
\]

Furthermore, \( t > \max\{T_1, T_2\} =: T_3 \) implies that

\[
\left| \frac{(Tx)(t)}{1 + \tau(t)} - \frac{\Phi^{-1}(A_x)}{a_+} \right|
\leq \frac{\epsilon}{2} + \int_0^t \frac{1}{\rho(s)} \left| \frac{\Phi^{-1}(A_x + \int_s^{+\infty} f(r, x(r), x'(r)) \, dr)}{a(s, x(s), x'(s))} - \frac{\Phi^{-1}(A_x)}{a_+} \right| \, ds
\]

\[
= \frac{\epsilon}{2} + \int_0^t \frac{1}{\rho(s)} \left| \frac{\Phi^{-1}(A_x + \int_s^{+\infty} f(r, x(r), x'(r)) \, dr)}{a(s, x(s), x'(s))} - \frac{\Phi^{-1}(A_x)}{a_+} \right| \, ds
\]

\[
+ \int_{T_3}^t \frac{1}{\rho(s)} \left| \frac{\Phi^{-1}(A_x + \int_s^{+\infty} f(r, x(r), x'(r)) \, dr)}{a(s, x(s), x'(s))} - \frac{\Phi^{-1}(A_x)}{a_+} \right| \, ds
\]

\[
\leq \frac{\epsilon}{2} + \int_0^t \frac{1}{\rho(s)} \, ds \left( \frac{\Phi^{-1}(M_1 + \int_s^{+\infty} \phi_M(r) \, dr)}{m} + \frac{\Phi^{-1}(M_1)}{a_+} \right) + \epsilon \int_{T_3}^t \frac{1}{\rho(s)} \, ds \frac{\Phi^{-1}(M_1)}{a_+}
\]

It is easy to see that there exists \( T_4 > T_3 \) such that

\[
\int_0^T \frac{1}{\rho(s)} \, ds \left( \frac{\Phi^{-1}(M_1 + \int_s^{+\infty} \phi_M(r) \, dr)}{m} + \frac{\Phi^{-1}(M_1)}{a_+} \right) < \epsilon, \quad t > T_4.
\]

Hence

\[
\left| \frac{(Tx)(t)}{1 + \tau(t)} - \frac{\Phi^{-1}(A_x)}{a_+} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = 2\epsilon, \quad t > T_4. \tag{2.12}
\]
So (2.11) and (2.12) imply that both \( \{ \rho(Tx)' : x \in D \} \) and \( \{ \frac{T_x}{1+r(t)} : x \in D \} \) are equi-convergent at \( +\infty \).

Similarly we can prove that both \( \{ \frac{T_x}{1+r(t)} : x \in D \} \) and \( \{ \rho(Tx)' : x \in D \} \) are equi-convergent at \( -\infty \). The details are omitted.

From Steps 3–5, we see that \( T \) maps bounded sets into relatively compact sets. Therefore, the operator \( T : X \to X \) is completely continuous. The proof is complete. \( \square \)

3 Main Theorems

In this section, the main results on the existence of solutions of BVP (1.7) are established.

Let \( L \) and \( L_n \) be defined in Section 1. For nonnegative functions \( a, b, c, a_1, b_1, c_1 \) and \( a_2, b_2, c_2 \in L^1(R) \), we denote

\[
\begin{align*}
\sigma_0 &= \frac{1}{m} + \frac{1}{m} \int_0^{+\infty} \alpha(s) \int_0^{s} \frac{1}{\rho(u)} du \, ds + \int_{-\infty}^{0} \alpha(s) \int_{0}^{s} \frac{1}{\rho(u)} du \, ds, \\
\Delta_1 &= \frac{\int_{-\infty}^{+\infty} [b_1(r) + c_1(r)] \, dr}{m \int_{-\infty}^{+\infty} \alpha(s) \, ds} + \frac{\int_{-\infty}^{+\infty} [b_2(r) + c_2(r)] \, dr}{1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} \, ds} \\
&\quad + \frac{\sigma_0 L_2 L_3 L \Phi^{-1}(2 \int_{-\infty}^{+\infty} b(r) \, dr)}{m} + \frac{\sigma_0 L_2 L_3 L \Phi^{-1}(2 \int_{-\infty}^{+\infty} c(r) \, dr)}{m}, \\
\Delta_2 &= \frac{\int_{-\infty}^{+\infty} [b_2(r) + c_2(r)] \, dr}{m(1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} \, ds)} \\
&\quad + \left( L_2 L_3 L \Phi^{-1}(2 \int_{-\infty}^{+\infty} b(r) \, dr) + L_2 L_3 L \Phi^{-1}(2 \int_{-\infty}^{+\infty} c(r) \, dr) \right) / m.
\end{align*}
\]

**Theorem 1.** Suppose that there exist nonnegative functions \( a, b, c, a_1, b_1, c_1 \) and \( a_2, b_2, c_2 \in L^1(R) \) satisfying \( \Delta_1 < 1 \), \( \Delta_2 < 1 \) and

\[
\begin{align*}
|f(t, (1 + \tau(t))x, \frac{1}{\rho(t)}y)| &\leq a(t) + b(t)\Phi(|x|) + c(t)\Phi(|y|), \quad x, y \in R, \ t \in R, \\
|g(t, (1 + \tau(t))x, \frac{1}{\rho(t)}y)| &\leq a_1(t) + b_1(t)|x| + c_1(t)|y|, \quad x, y \in R, \ t \in R, \\
|h(t, (1 + \tau(t))x, \frac{1}{\rho(t)}y)| &\leq a_2(t) + b_2(t)|x| + c_2(t)|y|, \quad x, y \in R, \ t \in R.
\end{align*}
\]

Then BVP (1.7) has at least one solution.

**Proof.** We will apply Lemma 1 to prove this theorem. Let \( X \) and \( T \) be defined in Section 2. From Lemma 3, \( T : X \to X \) is a completely continuous operator. Let

\[
M_5 = \max \left\{ \frac{\int_{-\infty}^{+\infty} a_1(r) \, dr}{\int_{-\infty}^{+\infty} \alpha(s) \, ds} + \sigma_0 L_2 \frac{\int_{-\infty}^{+\infty} a_2(r) \, dr}{1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} \, ds} \right\}
\]

Choose \( M_0 > M_5/(1 - \max\{\Delta_1, \Delta_2\}) \).

Now we define \( \Omega = \{x \in X: \|x\| < M_0\} \). We will show that \( T(\partial\Omega) \subset \overline{\Omega} \). In fact, if \( x \in \partial\Omega \), with \( \|Tx\| \geq M_0 \), then by definition of norm \( \|\cdot\| \) we have

\[
0 \leq \frac{|x(t)|}{1 + \tau(t)} \leq M_0, \quad \rho(t)|x''(t)| \leq M_0, \quad t \in R.
\]

By the definition of \( T \), together with (2.4), we get

\[
|B_x| = \left| \frac{\Phi^{-1}(A_x + \int_{-\infty}^{+\infty} f(r, x(r), x'(t)) dr) - \int_{-\infty}^{+\infty} g(r, x(r), x'(r)) dr}{\int_{-\infty}^{+\infty} \alpha(s) ds} \right|
\]

\[
- \frac{\int_{0}^{+\infty} \alpha(s) \int_{0}^{s} \frac{\Phi^{-1}(A_x + \int_{u}^{+\infty} f(r, x(r), x'(r)) dr)}{\rho(u) a(u, x(u), x'(u))} du ds}{\int_{-\infty}^{+\infty} \alpha(s) ds} \]

\[
+ \frac{\int_{-\infty}^{0} \alpha(s) \int_{s}^{0} \frac{\Phi^{-1}(A_x + \int_{u}^{+\infty} f(r, x(r), x'(r)) dr)}{\rho(u) a(u, x(u), x'(u))} du ds}{\int_{-\infty}^{+\infty} \alpha(s) ds} \]

\[
\leq \frac{\int_{-\infty}^{+\infty} |g(r, x(r), x'(r))| dr}{\int_{-\infty}^{+\infty} \alpha(s) ds} \]

\[
+ \frac{1 + \int_{0}^{+\infty} \alpha(s) \int_{0}^{s} \frac{1}{\rho(u)} du ds + \int_{-\infty}^{0} \alpha(s) \int_{s}^{0} \frac{1}{\rho(u)} du ds}{m \int_{-\infty}^{+\infty} \alpha(s) ds} \]

\[
\times \Phi^{-1}\left( \frac{\int_{-\infty}^{+\infty} |h(s, x(s), x'(s))| ds}{1 + \int_{-\infty}^{+\infty} \beta(s) ds} \right) + 2 \int_{-\infty}^{+\infty} f(r, x(r), x'(r)) dr \right).
\]

Then

\[
\sup_{t \in R} \left| Tx(t) \right| / (1 + \tau(t))
\]

\[
= \sup_{t \in R} \left\{ \begin{array}{ll}
\frac{B_x}{1 + \tau(t)} + \frac{\int_{0}^{t} \frac{\Phi^{-1}(A_x + \int_{u}^{+\infty} f(r, x(r), x'(r)) dr)}{\rho(s) a(s, x(s), x'(s))} du ds}{1 + \tau(t)}, & t \geq 0,
\frac{B_x}{1 + \tau(t)} - \frac{\int_{t}^{0} \frac{\Phi^{-1}(A_x + \int_{u}^{+\infty} f(r, x(r), x'(r)) dr)}{\rho(s) a(s, x(s), x'(s))} du ds}{1 + \tau(t)}, & t \leq 0,
\end{array} \right. \]

\[
\leq \sup_{t \in R} \left\{ \begin{array}{ll}
\frac{|B_x|}{1 + \tau(t)} + \frac{\int_{0}^{t} \frac{|\Phi^{-1}(A_x + \int_{u}^{+\infty} f(r, x(r), x'(r)) dr)|}{\rho(s) a(s, x(s), x'(s))} du ds}{1 + \tau(t)}, & t \geq 0,
\frac{|B_x|}{1 + \tau(t)} + \frac{\int_{t}^{0} \frac{|\Phi^{-1}(A_x + \int_{u}^{+\infty} f(r, x(r), x'(r)) dr)|}{\rho(s) a(s, x(s), x'(s))} du ds}{1 + \tau(t)}, & t \leq 0,
\end{array} \right. \]

\[
\begin{align*}
\sup_{t \in \mathbb{R}} & \left| B_x \right| + \frac{\int_t^t 1}{\rho(s)} \left| \Phi^{-1}(A_x + \int_a^b f(r, x(r), x'(r)) dr) \right| ds, \quad t \geq 0, \\
\left| B_x \right| & + \frac{\int_t^t 1}{\rho(s)} \left| \Phi^{-1}(A_x + \int_a^b f(r, x(r), x'(r)) dr) \right| ds, \quad t \leq 0,
\end{align*}
\]

\[
\begin{align*}
\sup_{t \in \mathbb{R}} & \left| B_x \right| + \frac{\int_t^t 1}{\rho(s)} \left| \Phi^{-1}(A_x + \int_{-\infty}^b f(r, x(r), x'(r)) dr) \right| ds, \quad t \geq 0, \\
\left| B_x \right| & + \frac{\int_t^t 1}{\rho(s)} \left| \Phi^{-1}(A_x + \int_{-\infty}^b f(r, x(r), x'(r)) dr) \right| ds, \quad t \leq 0,
\end{align*}
\]

\[
\begin{align*}
\left| B_x \right| & + \frac{\int_{-\infty}^{+\infty} |g(r, x(r), x'(r))| dr}{\int_{-\infty}^{+\infty} \alpha(s) ds} \\
& + \frac{\int_{-\infty}^{+\infty} \alpha(s) ds + \int_{-\infty}^{+\infty} \frac{1}{\rho(s)} \left| \Phi^{-1}(A_x + \int_{-\infty}^b f(r, x(r), x'(r)) dr) \right| ds}{\frac{1}{m} + \int_{-\infty}^{+\infty} \alpha(s) ds} \\
& \times \Phi^{-1} \left( f \left( \frac{\int_{-\infty}^{+\infty} h(s, x(s), x'(s)) ds}{1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} ds} \right) + 2 \int_{-\infty}^{+\infty} f(r, x(r), x'(r)) dr \right) \\
& = \int_{-\infty}^{+\infty} |g(r, x(r), x'(r))| dr + \int_{-\infty}^{+\infty} \alpha(s) ds \\
& + \sigma_0 \Phi^{-1} \left( f \left( \frac{\int_{-\infty}^{+\infty} h(s, x(s), x'(s)) ds}{1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} ds} \right) + 2 \int_{-\infty}^{+\infty} f(r, x(r), x'(r)) dr \right) \\
& \leq \int_{-\infty}^{+\infty} \left[ a_1(r) + b_1(r) \right] |x(r)| \frac{1}{1 + \tau(r)} + c_1(r) \rho(r) |x'(r)| ds \\
& + \sigma_0 L_2 \left[ \int_{-\infty}^{+\infty} a_2(r) |x(r)| \frac{1}{1 + \tau(r)} + c_2(r) \rho(r) |x'(r)| ds \right] \\
& + \sigma_0 L_2 \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} [a(r) + b(r) \Phi \left( \left| x(r) \right| \frac{1}{1 + \tau(r)} + c(r) \Phi (\rho(r) |x'(r)|) \right] dr \right) \\
& \leq \int_{-\infty}^{+\infty} \left[ a_1(r) + b_1(r) \right] |x| + c_1(r) \rho(r) |x| ds \\
& + \sigma_0 L_2 \left[ \int_{-\infty}^{+\infty} a_2(r) |x| + c_2(r) \rho(r) |x| \right] ds \left( 1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} ds \right)
\end{align*}
\]
It follows that
\[
\sup_t |T x(t)| \leq \int_{-\infty}^{+\infty} a_1(r) dr + \sigma_0 L_2 \int_{-\infty}^{+\infty} a_2(r) dr \left( 1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} ds \right) 
\]
\[
+ \sigma_0 L_2 \int_{-\infty}^{+\infty} \left[ b_1(r) + c_1(r) \right] ds + \sigma_0 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} b(r) dr \right) 
\]
\[
+ \sigma_0 L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} c(r) dr \right) \right] \parallel x \parallel.
\]

It follows that
\[
\sup_{t \in \mathbb{R}} |(T x)'(t)| \leq \int_{-\infty}^{+\infty} a_1(r) dr + \sigma_0 L_2 \int_{-\infty}^{+\infty} a_2(r) dr \left( 1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} ds \right) 
\]
\[
+ \sigma_0 L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} a(r) dr \right) + \Delta_1 M_0.
\]

Similarly, we have
\[
\sup_{t \in \mathbb{R}} \rho(t) |(T x)'(t)| \leq \sup_{t \in \mathbb{R}} \frac{|\Phi^{-1}(A x + \int_{t}^{+\infty} f(r, x(r), x'(r)) dr)|}{a(t, x(t), x'(t))} 
\]
\[
\leq \left( L_2 \int_{-\infty}^{+\infty} a_2(r) dr \left( 1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} ds \right) + L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} a(r) dr \right) / m 
\]
\[
+ \frac{\parallel x \parallel}{m} \left[ L_2 \int_{-\infty}^{+\infty} \left[ b_2(r) + c_2(r) \right] dr \left( 1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} ds \right) 
\]
\[
+ L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} b(r) dr \right) + L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} c(r) dr \right) \right].
\]

It follows that
\[
\sup_{t \in \mathbb{R}} \rho(t) |(T x)'(t)| \leq L_2 \frac{\int_{-\infty}^{+\infty} a_2(r) dr}{m(1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} ds)} 
\]
\[
+ \frac{L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} a(r) dr \right)}{m} + \Delta_2 M_0.
\]
Then (3.2) and (3.3) imply that
\[ \|Tx\| \leq M_5 + \max\{\Delta_1, \Delta_2\} M_0. \] (3.4)
It follows from \( \|Tx\| \geq M_0 \) that
\[ M_0 \leq M_5 / \left(1 - \max\{\Delta_1, \Delta_2\}\right), \]
a contradiction to (3.1). So \( T(\partial\Omega) \subset \overline{\Omega} \). Thus Lemma 1 implies that the operator \( T \) has at least one fixed point in \( \Omega \). So BVP (1.7) has at least one solution. \( \square \)

**Corollary 1.** Suppose that there exists \( r > 0 \) such that
\[
\int_{-\infty}^{+\infty} |g\left(t, (1 + \tau(t)) x, \frac{1}{\rho(t)} y\right)| \, dt \leq \frac{r}{3} \int_{-\infty}^{+\infty} \alpha(s) \, ds,
\]
\[
\int_{-\infty}^{+\infty} |h\left(t, (1 + \tau(t)) x, \frac{1}{\rho(t)} y\right)| \, dt \leq \frac{r}{3\sigma_0 L_2} \left(1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} \, ds\right),
\]
\[
\int_{-\infty}^{+\infty} |f\left(t, (1 + \tau(t)) x, \frac{1}{\rho(t)} y\right)| \, dt \leq \frac{1}{2} \Phi\left(\frac{r}{3\sigma_0 L_2}\right),
\]
where \( x, y \in [-r, r] \). Then BVP (1.7) has at least one solution.

**Proof.** From Lemma 3, \( T : X \rightarrow X \) is a completely continuous operator. Now we define \( \Omega = \{x \in X : \|x\| < r\} \). For any \( x \in \partial\Omega, \|x\| = r \). So
\[
\sup_{t \in \mathbb{R}} \frac{|x(t)|}{1 + \tau(t)} \leq r, \quad \sup_{t \in \mathbb{R}} \rho(t) |x'(t)| \leq r.
\]
By the assumptions, similarly to the proof of Theorem 1, we get
\[
\sup_{t \in \mathbb{R}} \frac{|Tx(t)|}{1 + \tau(t)} \leq \frac{\int_{-\infty}^{+\infty} |g(r, x(r), x'(r))| \, dr}{\int_{-\infty}^{+\infty} \alpha(s) \, ds}\]
\[+ \sigma_0 L_2 \left[\frac{\int_{-\infty}^{+\infty} |h(s, x(s), x'(s))| \, ds}{1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} \, ds} + \Phi^{-1}\left(2 \int_{-\infty}^{+\infty} f(r, x(r), x'(r)) \, dr\right)\right]\]
\[\leq \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r = \|x\|.
\]
Furthermore,
\[
\sup_{t \in \mathbb{R}} \rho(t) |(Tx)'(t)| \leq \frac{1}{m} \left(L_2 \frac{\int_{-\infty}^{+\infty} |h(s, x(s), x'(s))| \, ds}{1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M \rho(s)} \, ds} + L_2 \Phi^{-1}\left(2 \int_{-\infty}^{+\infty} |f(r, x(r), x'(r))| \, dr\right)\right)
\]
\[
\leq \frac{1}{m3\sigma_0L_2}\left(\frac{r(1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M\rho(s)} \, ds)}{1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M\rho(s)} \, ds} + L_2r\right) = \frac{2r}{3m\sigma_0} < r = \|x\|.
\]

So \(\|Tx\| \leq \|x\|\) for all \(x \in \partial\Omega\). Similar to the process in Theorem 1, the result follows. The proof is complete. □

**Corollary 2.** Suppose that
\[
\lim_{d \to +\infty} \frac{\max_{x,y \in [-d,d]} \int_{-\infty}^{+\infty} |f(s,(1 + \tau(s))x, \frac{1}{\rho(s)}y)| \, ds}{\Phi(d)} = 0,
\]
\[
\lim_{d \to +\infty} \frac{\max_{x,y \in [-d,d]} \int_{-\infty}^{+\infty} |g(s,(1 + \tau(s))x, \frac{1}{\rho(s)}y)| \, ds}{d} = 0,
\]
\[
\lim_{d \to +\infty} \frac{\max_{x,y \in [-d,d]} \int_{-\infty}^{+\infty} |h(s,(1 + \tau(s))x, \frac{1}{\rho(s)}y)| \, ds}{d} = 0.
\]

Then BVP (1.7) has at least one solution.

**Proof.** Let
\[
\varepsilon = \min\left\{ \frac{1}{3} \int_{-\infty}^{+\infty} \alpha(s) \, ds, \frac{1}{3\sigma_0L_2}\left(1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M\rho(s)} \, ds\right), \frac{1}{2}\Phi\left(\frac{1}{3\sigma_0L_2}\right) \right\}.
\]

Then, there exists \(r > 0\), such that
\[
\int_{-\infty}^{+\infty} \left| g\left(s,(1 + \tau(s))x, \frac{1}{\rho(s)}y\right) \right| \, ds \leq \frac{r}{3} \int_{-\infty}^{+\infty} \alpha(s) \, ds,
\]
\[
\int_{-\infty}^{+\infty} \left| h\left(s,(1 + \tau(s))x, \frac{1}{\rho(s)}y\right) \right| \, ds \leq \frac{r}{3\sigma_0L_2}\left(1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M\rho(s)} \, ds\right),
\]
\[
\int_{-\infty}^{+\infty} \left| f\left(s,(1 + \tau(s))x, \frac{1}{\rho(s)}y\right) \right| \, ds \leq \frac{1}{2}\Phi\left(\frac{r}{3\sigma_0L_2}\right).
\]

By Corollary 1, BVP (1.7) has at least one solution. The proof is complete. □

**4 An Example**

Now, we present an example to illustrate Theorem 1.

**Example 1.** Consider the following problem
\[
\left[ \Phi(e^{-|t|}a(t,x(t),x'(t))x'(t)) \right]' + \lambda e^{-t^2} + \frac{1}{1 + t^2}\left(1 + \int_0^t e^{|s|} \, ds\right)^{-3} [x(t)]^3
\]
\[
+ \frac{|t|}{1 + t^4} e^{-3|t|} [x'(t)]^3 = 0, \quad t \in R,
\]
\[
\lim_{t \to -\infty} e^t a(t,x(t),x'(t))x'(t) - \int_{-\infty}^{+\infty} e^{-2|s|} x(s) \, ds = 0,
\]
\[
\lim_{t \to +\infty} e^{-t} a(t,x(t),x'(t))x'(t) + \int_{-\infty}^{+\infty} e^{-2|s|} x'(s) \, ds = 0,
\]
(4.1)
where \( \lambda \in R \) is a constant, \( \Phi(x) = |x|^2 x \) is a one-dimensional \( p \)-Lapacian. Then BVP (4.1) has at least one solution if

\[
|\lambda| < \frac{8}{27\pi(\sqrt{2} + 1)^3}.
\]

**Proof.** Corresponding to BVP (1.7), we have \( \Phi(x) = |x|^2 x, \rho(t) = e^{-|t|}, \) with

\[
\tau(t) = \int_0^t \frac{ds}{\rho(s)} = \begin{cases} e^{t} - 1, & t \geq 0, \\ 1 - e^{-t}, & t \leq 0, \end{cases}
\]

\[
a(t, x, y) = 2 + \frac{x^2}{(1+\tau(t))^3 + x^2} + \frac{y^2}{e^{6|t|} + y^2}, \quad \alpha(t) = \beta(t) = e^{-2|t|},
\]

\[
f(t, x, y) = \lambda \left[ e^{-t^2} + \frac{1}{1 + t^2} \left( 1 + \left| \int_0^t e^{s^2} ds \right| \right)^3 x^3 + \frac{|t|}{1 + t^4} e^{-3|t|} y^3 \right],
\]

and \( g(t, x, y) = h(t, x, y) \equiv 0. \)

One can show that

- \( \rho \in C^0(R, [0, +\infty)) \) with \( \rho(t) > 0 \) for all \( t \in R \) satisfies

\[
\int_{-\infty}^0 \frac{1}{\rho(s)} ds = +\infty, \quad \int_0^{+\infty} \frac{1}{\rho(s)} ds = +\infty.
\]

We find that

- \( a : R \times R \times R \to (0, +\infty) \) is continuous and satisfies

\[
2 \leq a(t, (1+\tau(t))x, y/\rho(t)) \leq 4, \quad t \in R, \ x \in R, \ y \in R
\]

and for each \( r > 0 \), \( |x|, |y| \leq r \) imply that

\[
a(t, (1+\tau(t))x, y/\rho(t)) = 2 + \frac{x^2}{(1+\tau(t))^3 + x^2} + \frac{y^2}{e^{3|t|} + y^2} \to a_{+\infty} = 2
\]

uniformly as \( t \to \pm\infty. \)

- \( \alpha, \beta : R \to [0, +\infty) \) are continuous functions satisfying

\[
\int_{-\infty}^{+\infty} \alpha(s) ds > 0, \quad \int_0^{+\infty} \alpha(s) \int_0^s \frac{dr}{\rho(r)} ds < +\infty,
\]

\[
\int_{-\infty}^{0} \alpha(s) \int_s^{0} \frac{dr}{\rho(r)} ds < +\infty, \quad \int_{-\infty}^{+\infty} \beta(s) \int_{s}^{0} \frac{dr}{\rho(r)} ds < +\infty.
\]

It is well known that \( (s + t)^{\frac{1}{2}} \leq s^\frac{1}{2} + t^\frac{1}{2} \) for all \( s, t \geq 0. \)

- \( \Phi(x) = |x|^2 x, \) is continuous and strictly increasing on \( R, \Phi(0) = 0 \) and its inverse function is \( \Phi^{-1}(x) = |x|^{-\frac{2}{3}} x \) for \( x \neq 0 \) and \( \Phi^{-1}(0) = 0 \) is continuous too, moreover \( \Phi^{-1} \) satisfies that there exist constants \( L > 0 \) and \( L_n > 0 \) such that \( \Phi^{-1}(x_1 x_2) \leq L \Phi^{-1}(x_1) \Phi^{-1}(x_2) \) with \( L = 1 \) and

\[
\Phi^{-1}(x_1 + \cdots + x_n) \leq L_n \left[ \Phi^{-1}(x_1) + \cdots + \Phi^{-1}(x_n) \right]
\]

holds for all \( x_i \geq 0 \) \( (i = 1, 2, \ldots, n) \) with \( L_n = 1. \)
• $f, g, h$ defined on $R^3$ are nonnegative Caratheodory functions. To apply Theorem 1, choose

$$ a(t) = |\lambda|e^{-t^2}, \quad b(t) = |\lambda|\frac{1}{1+t^2}, \quad c(t) = |\lambda|\frac{|t|}{1+t^4}, $$

$$ a_1(t) = a_2(t) = b_1(t) = b_2(t) = c_1(t) = c_2(t) = 0. $$

It is easy to show that $a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 \in L^1(R)$ and

$$ |f(t, (1+\tau(t))x, \frac{1}{\rho(t)}y)| \leq a(t) + b(t)\Phi(|x|) + c(t)\Phi(|y|), \quad x, y \in R, \ t \in R, $$

$$ |g(t, (1+\tau(t))x, \frac{1}{\rho(t)}y)| \leq a_1(t) + b_1(t)|x| + c_1(t)|y|, \quad x, y \in R, \ t \in R, $$

$$ |h(t, (1+\tau(t))x, \frac{1}{\rho(t)}y)| \leq a_2(t) + b_2(t)|x| + c_2(t)|y|, \quad x, y \in R, \ t \in R. $$

By direct computation, we get

$$ \sigma_0 = \frac{1}{m} + \frac{1 + \int_0^{+\infty} \alpha(s) \int_0^{s} \frac{1}{\rho(u)} du \ ds + \int_{-\infty}^{0} \alpha(s) \int_{s}^{0} \frac{1}{\rho(u)} du \ ds}{\int_{-\infty}^{+\infty} \alpha(s) ds} = \frac{3}{2}, $$

$$ \Delta_1 = \frac{\int_{-\infty}^{+\infty} [b_1(r) + c_1(r)] dr}{\int_{-\infty}^{+\infty} \alpha(s) ds} + \sigma_0 L_2 \frac{\int_{-\infty}^{+\infty} [b_2(r) + c_2(r)] dr}{1 + \int_{-\infty}^{+\infty} \frac{\rho(s)}{M\rho(s)} ds} $$

$$ + \sigma_0 L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} b(r) \ dr \right) + \sigma_0 L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} c(r) \ dr \right) $$

$$ = \frac{3}{2} |\lambda|^\frac{1}{4} ([2\pi]^\frac{1}{2} + \pi^\frac{1}{2}), $$

$$ \Delta_2 = L_2 \frac{\int_{-\infty}^{+\infty} [b_2(r) + c_2(r)] dr}{m(1 + \int_{-\infty}^{+\infty} \frac{\beta(s)}{M\rho(s)} ds)} $$

$$ + \frac{L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} b(r) \ dr \right) + L_2 L_3 L \Phi^{-1} \left( 2 \int_{-\infty}^{+\infty} c(r) \ dr \right)}{m} $$

$$ = \frac{1}{2} |\lambda|^\frac{1}{4} ([2\pi]^\frac{1}{2} + \pi^\frac{1}{2}). $$

It follows from Theorem 1 that BVP (4.1) has at least one solution if

$$ |\lambda| < \frac{8}{27\pi(\sqrt{2} + 1)^3}. \quad \Box $$

**Acknowledgments**

The author thanks Alessandro Calamai for his great help in improving the exposition of the paper. The author also thanks the anonymous referees for their careful reading of this manuscript and for suggesting some useful stylistic changes.
References
