Joint Universality of Dirichlet $L$-Functions and Periodic Hurwitz Zeta-Functions$^*$

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Received June 29, 2012; revised September 20, 2012; published online November 1, 2012

Abstract. In the paper, we prove that every system of analytic functions can be approximated simultaneously uniformly on compact subsets of some region by a collection consisting of shifts of Dirichlet $L$-functions with pairwise non-equivalent characters and periodic Hurwitz zeta-functions with parameters algebraically independent over the field of rational numbers.

Keywords: Dirichlet $L$-function, limit theorem, periodic Hurwitz zeta-function, space of analytic functions, universality.

AMS Subject Classification: 11M06; 11M41.

1 Introduction

Since a remarkable Voronin’s work [24] on the universality of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, it is known that the majority of other zeta and $L$-functions also are universal in the sense that their shifts approximate uniformly on compact subsets of certain regions wide classes of analytic functions, for results and references, see [1, 4, 6, 7, 12, 17, 22]. Also, a more complicated approximation property of zeta and $L$-functions – the joint universality – is known. In this case, we deal with a simultaneous approximation of a given system of analytic functions. The first result in this direction also is due to Voronin who obtained [23] the joint universality of Dirichlet $L$-functions. The

$^*$ The third author is partially supported by the project LYMOS (1.2.1 activity), No. VPI-3.2-ŠMM-02-V-02-001.
joint universality for Hurwitz zeta-functions was proved in [20] and [9]. We observe that Dirichlet L-functions have Euler product over primes while Hurwitz zeta-functions \( \zeta(s, \alpha) \), \( 0 < \alpha \leq 1 \), do not have Euler product, except for the cases \( \zeta(s, 1) = \zeta(s) \) and \( \zeta(s, 1/2) = (2^s - 1) \zeta(s) \).

In [19] H. Mishou began to study the joint universality for zeta-functions having and having no Euler product over primes. He proved a joint universality theorem for the Riemann zeta-function and Hurwitz zeta-function with transcendental parameter \( \alpha \).

Let \( D = \{ s \in \mathbb{C} : 1/2 < \sigma < 1 \} \). Denote by \( K \) the class of compact subsets of \( D \) with connected complements. Moreover, let \( \mathcal{H}_0(K) \) and \( \mathcal{H}(K) \), \( K \in K \), be the classes of continuous non-vanishing and continuous on \( K \) functions, respectively, which are analytic in the interior of \( K \). Denote by \( \text{meas}\{ A \} \) the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \). Then the Mishou theorem is stated as follows: Suppose that \( \alpha \) is transcendental. Let \( K_1, K_2 \in K \), and \( f_1 \in \mathcal{H}_0(K_1) \), \( f_2 \in \mathcal{H}(K_2) \). Then, for every \( \varepsilon > 0 \),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \right\} > 0.
\]

We call a property of \( \zeta(s) \) and \( \zeta(s, \alpha) \) in the later theorem the mixed joint universality.

In [5], the Mishou theorem was generalised for a periodic zeta-function and a periodic Hurwitz zeta-function. Let \( a = \{ a_m : m \in \mathbb{N} \} \) and \( b = \{ b_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \} \) be two periodic sequences of complex numbers. Then the periodic zeta-function \( \zeta(s; a) \) and the periodic Hurwitz zeta-function \( \zeta(s, \alpha; b) \) are defined, for \( \sigma > 1 \), by

\[
\zeta(s; a) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \quad \text{and} \quad \zeta(s, \alpha; b) = \sum_{m=0}^{\infty} \frac{b_m}{(m + \alpha)^s},
\]

respectively, and by analytic continuation elsewhere, except for a possible poles at \( s = 1 \).

In [10] a mixed universality theorem was extended to a collection consisting of several periodic zeta-functions and periodic Hurwitz zeta-functions.

In the case of periodic Hurwitz zeta-functions, the following more general joint universality can be considered. For \( j = 1, \ldots, r \), \( \alpha_j \), let \( 0 < \alpha_j \leq 1 \) be a fixed parameter, \( l_j \in \mathbb{N} \), \( a_{jl} = \{ a_{mj} : m \in \mathbb{N}_0 \} \) be a periodic sequence of complex numbers with minimal period \( k_{jl} \in \mathbb{N} \), and \( \zeta(s, \alpha_j; a_{jl}) \) be the corresponding periodic Hurwitz zeta-function. In [8, 14, 15], the joint universality for the functions

\[
\zeta(s, \alpha_1; a_{11}), \ldots, \zeta(s, \alpha_1; a_{1l_1}), \ldots, \zeta(s, \alpha_r; a_{r1}), \ldots, \zeta(s, \alpha_r; a_{rl_r}) \tag{1.1}
\]

was obtained. Later, the mixed joint universality for system (1.1) extended by some zeta-functions having Euler product was began to study. In [3], the Riemann zeta-function was added to the system (1.1). In the subsequent papers
[13, 16, 21], the function $\zeta(s)$ was replaced by zeta-functions of certain cusp forms. Namely, the paper [21] is devoted to a mixed joint universality theorem for the zeta-function $\zeta(s, F)$ attached to a normalized Hecke eigen cusp form $F$ and the functions (1.1), in [16], the function $\zeta(s, F)$ was replaced by a zeta-function of a new form, and in [13], the case of a zeta-function of a cusp form $F$ with respect to the Hecke subgroup with Dirichlet character was considered.

The aim of this paper is to extend the system (1.1) by a collection of Dirichlet $L$-functions. The extension of the class of jointly universal functions is motivated by wide theoretical and practical applications of universality (functional independence, zero-distribution, various value denseness problems, approximation and estimation of complicated analytic functions and their functionals).

Let $\chi$ be a Dirichlet character modulo $q$. We remind that the corresponding Dirichlet $L$-function $L(s, \chi)$ is defined, for $\sigma > 1$, by the series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and is analytically continued to an entire function provided $\chi$ is a non-principal character. For the principal character $\chi_0$,

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Let $k_j = [k_{j1}, \ldots, k_{jlj}]$ be the least common multiple of the periods $k_{j1}, \ldots, k_{jlj}$, and

$$A_j = \begin{pmatrix}
    a_{1j1} & a_{1j2} & \cdots & a_{1jlj} \\
    a_{2j1} & a_{2j2} & \cdots & a_{2jlj} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{kj1} & a_{kj2} & \cdots & a_{kjlj}
\end{pmatrix}, \quad j = 1, \ldots, r.$$

The main result of the paper is contained in the following theorem.

**Theorem 1.** Suppose that $\chi_1, \ldots, \chi_d$ are pairwise non-equivalent Dirichlet characters, the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over the field of rational numbers, and that $\text{rank}(A_j) = l_j$, $j = 1, \ldots, r$. For $j = 1, \ldots, d$, let $K_j \in \mathcal{K}$ and $f_j \in \mathcal{H}_0(K_j)$, and, for $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, let $K_{jl} \in \mathcal{K}$ and $f_{jl} \in \mathcal{H}(K_{jl})$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T]: \sup_{1 \leq j \leq d} \sup_{s \in K_j} \left|L(s + i\tau, \chi_j) - f_j(s)\right| < \varepsilon,\right.$$

$$\sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} \left|\zeta(s + i\tau, \alpha_j; a_{jl}) - f_{jl}(s)\right| < \varepsilon\left\} > 0.$$

### 2 Multidimensional Limit Theorem

The main ingredient of the proof of Theorem 1 is a limit theorem for probability measures in the multidimensional space of analytic functions.
Denote by $H(D)$ the space of analytic functions on $D$ endowed with the topology of uniform convergence on compacta, and let

$$H_{d,u} = H_{d,u}(D) = H^d(D) \times H^u(D), \quad u = \sum_{j=1}^{r} l_j.$$ 

Let $\mathcal{B}(S)$ stand for the $\sigma$-field of Borel sets of the space $S$. This section is devoted to weak convergence of probability measures defined by terms of Dirichlet $L$-functions and periodic Hurwitz zeta-functions in the space $(H_{d,u}, \mathcal{B}(H_{d,u}))$.

Let $\gamma = \{s \in \mathbb{C}: |s| = 1\}$ be the unit circle on the complex plane, and

$$\Omega = \prod_{p} \gamma_p, \quad \Omega_1 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes $p$, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. The tori $\Omega$ and $\Omega_1$ are compact topological Abelian groups. Define

$$\Omega^\kappa = \Omega \times \Omega_1 \times \cdots \times \Omega_{1r},$$

where $\Omega_{1j} = \Omega_1$, for all $j = 1, \ldots, r$, and $\kappa = 1 + r$. Then $\Omega^\kappa$ also is a compact topological group. Therefore on $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa))$, the probability Haar measure $m_H^\kappa$ can be defined. This gives the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$. Let

$$\Omega_{1r}^\kappa = \Omega_{11} \times \cdots \times \Omega_{1r},$$

Then we have that the measure $m_H^\kappa$ is the product of the probability Haar measures $m_H$ and $m_H^\kappa$ on $(\Omega, \mathcal{B}(\Omega))$ and $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa))$, respectively.

Now, on the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$, define a $H_{d,u}$-valued random element. We denote by $\omega_p$ the projection of $\omega \in \Omega$ to $\gamma_p$, and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_{1j}$ to $\gamma_m$. Let, for brevity, $\omega = (\omega, \omega_1, \ldots, \omega_r)$, $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\chi = (\chi_1, \ldots, \chi_d)$ and $\mathfrak{a} = (a_{11}, \ldots, a_{1l_1}, \ldots, a_{r1}, \ldots, a_{rl_r})$. Let the $H_{d,u}$-valued random element $F(s, \chi, \alpha, \omega; \mathfrak{a})$ be given by

$$F(s, \chi, \alpha, \omega; \mathfrak{a}) = \left( L(s, \omega, \chi_1), \ldots, L(s, \omega, \chi_d), \zeta(s, \alpha_1, \omega_1; a_{11}) \ldots, \zeta(s, \alpha_r, \omega_r; a_{rl_r}) \right),$$

where

$$L(s, \omega, \chi_j) = \prod_{p} \left( 1 - \frac{\chi_j(p)}{p^s} \right)^{-1}, \quad j = 1, \ldots, d,$$

and

$$\zeta(s, \alpha_j, \omega_j; a_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}(\omega_j(m))}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j.$$ 

Denote by $P_F$ the distribution of the random element $F(s, \chi, \alpha, \omega; \mathfrak{a})$, i.e., for $A \in \mathcal{B}(H_{d,u})$,

$$P_F(A) = m^\kappa_H \left( \omega \in \Omega^\kappa : F(s, \chi, \alpha, \omega; \mathfrak{a}) \in A \right).$$

Moreover,

$$F(s, \chi, \alpha; \mathfrak{a}) = \left( L(s, \chi_1), \ldots, L(s, \chi_d), \zeta(s, \alpha_1; a_{11}) \ldots, \zeta(s, \alpha_r; a_{rl_r}) \right).$$
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**Theorem 2.** Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$. Then

$$P_T(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\{\tau \in [0, T] : F(s + i\tau, \chi, \alpha; a) \in A\}, \quad A \in B(H_{d,u}),$$

converges weakly to $P_F$ as $T \to \infty$.

Limit theorems for as a wide system of functions as in Theorem 2 are not known, however, their proofs differ from that, for example, in [4] only by details. Therefore, we present a shortened proof of Theorem 2. Denote by $P$ the set of all prime numbers.

**Lemma 1.** Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then

$$Q_T(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\{\tau \in [0, T] : ((p^{-i\tau}: p \in P), ((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \ldots, ((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0)) \in A\}, \quad A \in B(\mathbb{Q}^\kappa),$$

converges weakly to the Haar measure $m^\kappa_H$ as $T \to \infty$.

Proof of the lemma is given in [10, Theorem 3].

Let $\sigma_0 > \frac{1}{2}$ be a fixed number, and

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_0}\right\}, \quad v_n(m, \alpha) = \exp\left\{-\left(\frac{m + \alpha}{n + \alpha}\right)^{\sigma_0}\right\}.$$

Define

$$L_n(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)v_n(m)}{m^s}, \quad j = 1, \ldots, d,$$

$$\zeta_n(s, \alpha_j; a_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j,$$

and, for $\omega_0 = (\omega_0, \omega_{10}, \ldots, \omega_{r0}) \in \mathbb{Q}^\kappa$,

$$L_n(s, \chi_j, \omega_0) = \sum_{m=1}^{\infty} \frac{\chi_j(m)v_n(m)\omega_0(m)}{m^s}, \quad j = 1, \ldots, d,$$

$$\zeta_n(s, \alpha_j, \omega_{0j}; a_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_0j(m)v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j.$$

All later series are absolutely convergent for $\sigma > \frac{1}{2}$. Here $\omega(p)$ is extended to the set $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{\substack{p^k|m \\text{ or } p^{k+1}|m}} \omega^k(p), \quad m \in \mathbb{N}.$$

Let, for brevity,

\[ F_n(s, \chi; \alpha; a) = (L_n(s, \chi_1), \ldots, L_n(s, \chi_d), \zeta_n(s, \alpha_1; a_{11}), \ldots, \zeta_n(s, \alpha_1; a_{11}), \ldots, \zeta_n(s, \alpha_r; a_{r1}), \ldots, \zeta_n(s, \alpha_r; a_{r1})), \]

and

\[ F_n(s, \chi; \alpha, \omega; a) = (L_n(s, \omega, \chi_1), \ldots, L_n(s, \omega, \chi_d), \zeta_n(s, \alpha_1, \omega_1; a_{11}), \ldots, \zeta_n(s, \alpha_1, \omega_1; a_{11}), \ldots, \zeta_n(s, \alpha_r, \omega_r; a_{r1}), \ldots, \zeta_n(s, \alpha_r, \omega_r; a_{r1})). \]

**Lemma 2.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \). Then on \((H_{n,r}, B(H_{n,r}))\), there exists a probability measure \( P_n \) such that

\[
P_{T,n}(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\{ \tau \in [0, T]: F_n(s + i\tau, \chi; \alpha; a) \in A \}, \quad A \in B(H_{d,u}),
\]

\[
\hat{P}_{T,n}(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\{ \tau \in [0, T]: F_n(s + i\tau, \chi; \alpha; \omega; a) \in A \}, \quad A \in B(H_{d,u}),
\]

both converge weakly to \( P_n \) as \( T \to \infty \).

**Proof.** Define \( h_n : \Omega^\kappa \to H_{d,r} \) by the formula \( h_n(\omega) = F_n(s, \chi; \alpha; \omega; a) \). In view of the absolute convergence of the series for \( L_n(s, \chi_j, \omega) \) and \( \zeta_n(s, \alpha_j, \omega_j; a_{jl}) \), the function \( h_n \) is continuous, and

\[
h_n((p^{-i\tau}: p \in \mathcal{P}), ((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \ldots, ((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0)) = F_n(s + i\tau, \chi; \alpha; a).
\]

Thus, we have that \( P_{T,n} = Q_T h_n^{-1} \). This, continuity of \( h_n \), and Lemma 1 show that \( P_{T,n} \) converges weakly to \( P_n = m_T^\kappa h_n^{-1} \) as \( T \to \infty \).

The invariance of the Haar measure \( m_T^\kappa \) allows to prove that the measure \( \hat{P}_{T,n} \) also converges weakly to \( P_n \). \( \square \)

Define

\[
\hat{P}_T(A) = \frac{1}{T} \text{meas}\{ \tau \in [0, T]: F(s + i\tau, \chi; \alpha, \omega; a) \in A \}, \quad A \in B(H_{d,u}).
\]

**Lemma 3.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \). Then on \((H_{d,u}, B(H_{d,u}))\), there exists a probability measure \( P \) such that \( P_T \) and \( \hat{P}_T \) both converge weakly to \( P \) as \( T \to \infty \).

**Proof.** To prove the lemma it suffices to pass from \( F_n(s, \chi; \alpha; a) \) and \( F_n(s, \chi; \alpha; \omega; a) \) to \( F(s, \chi; \alpha; a) \) and \( F(s, \chi; \alpha; \omega; a) \), respectively. For this, define a metric on \( H_{d,u} \) which induces the topology of uniform convergence on compacta. Let \( \{K_v : v \in \mathbb{N}\} \subset D \) be a sequences on compact subsets such that \( D = \bigcup_{v=1}^\infty K_v \), \( K_l \subset K_{l+1}, l \in \mathbb{N}, \) and if \( K \subset D \) is a compact subset, then \( K \subset K_v \), for some \( v \).

For \( g_1, g_2 \in H(D) \), define

\[
\rho(g_1, g_2) = \sum_{v=1}^\infty 2^{-v} \sup_{s \in K_v} |g_1(s) - g_2(s)| \frac{1}{1 + \sup_{s \in K_v} |g_1(s) - g_2(s)|}.
\]
Then $\rho$ is a metric on $H(D)$ inducing the topology of uniform convergence on compacta. For $g_j = (g_{j1}, \ldots, g_{jd+u}) \in H_{d,u}$, $j = 1, 2$, we put

$$\rho_{d+u}(g_1, g_2) = \max_{1 \leq i \leq d+u} \rho(g_{1i}, g_{2i}).$$

Then $\rho_{d+u}$ is a desired metric on $H_{d,u}$. Let $\rho_d$ and $\rho_r$ be analogical metrics on $H^d(D)$ and $H^u(D)$, respectively. We put

$$L_n(s, \chi) = (L_n(s, \chi_1), \ldots, L_n(s, \chi_d)), \quad \zeta_n(s, \alpha; a) = (\zeta(s, \alpha_1; a_{11}), \ldots, \zeta(s, \alpha_1; a_{1l_1}), \ldots, \zeta(s, \alpha_r; a_{rl}), \ldots, \zeta(s, \alpha_r; a_{rl})).$$

and

$$L_n(s, \chi) = (L_n(s, \chi_1), \ldots, L_n(s, \chi_d)), \quad \zeta_n(s, \alpha; a) = (\zeta_n(s, \alpha_1; a_{11}), \ldots, \zeta_n(s, \alpha_r; a_{rl}), \ldots, \zeta_n(s, \alpha_r; a_{rl})).$$

Then, from the proof of a limit theorem for Dirichlet $L$-functions in [11], it follows that

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_d(L_n(s + i\tau, \chi), L_n(s + i\tau, \chi)) \, d\tau = 0.$$

Similarly, in [8], it was obtained that

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_r(\zeta(s + i\tau, \alpha; a), \zeta_n(s + i\tau, \alpha; a)) \, d\tau = 0.$$

Two last equalities together with the definition of the metric $\rho_r$ show that

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_{d+u}(F(s + i\tau, \chi, \alpha; a), F_n(s + i\tau, \chi, \alpha; a)) \, d\tau = 0. \quad (2.1)$$

Now let $\theta$ be a random variable defined on a certain probability space $(\hat{\Omega}, \bar{\mathcal{A}}, \bar{\mathcal{P}})$ and uniformly distributed on $[0, 1]$. Define the $H_{d,u}$-valued random element $X_{T,n} = X_{T,n}(s) = (X_{T,n,1}(s), \ldots, X_{T,n,d}(s), \hat{X}_{T,n,1,1}(s), \ldots, \hat{X}_{T,n,1,l_1}(s), \ldots, \hat{X}_{T,n,r,1}(s), \ldots, \hat{X}_{T,n,r,l_r}(s)) = F_n(s + i\theta T, \chi, \alpha; a)$, and denote by $\overset{D}{\to}$ the convergence in distribution. Then, by Lemma 2, we have that

$$X_{T,n} \overset{D}{\to} X_n, \quad (2.2)$$

where $X_n = (X_{n,1}, \ldots, X_{n,d}, \hat{X}_{n,1,1}, \ldots, \hat{X}_{n,1,l_1}, \ldots, \hat{X}_{n,r,1}, \ldots, \hat{X}_{n,r,l_r})$ is a $H_{d,r}$-valued random element having the distribution $P_n$ ($P_n$ is the limit measure in Lemma 2). We will prove that the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight.

Since the series for $L_n(s, \chi_j)$ converges absolutely for $\sigma > \frac{1}{2}$, we find that

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K_v} |L_n(s + i\tau, \chi_j)| \, d\tau \leq C_v R_j v, \quad (2.3)$$

where

\[ R_{jv} = \left( \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma_v}} \right)^{\frac{1}{2}}, \quad j = 1, \ldots, d, \; l \in \mathbb{N}, \]

with some \( C_v > 0 \) and \( \sigma_v > \frac{1}{2} \). Similarly, the absolute convergence for \( \zeta_n(s, \alpha_j; a_{jl}) \) leads to the estimate

\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K_v} \left| \zeta_n(s + i\tau, \alpha_j; a_{jl}) \right| d\tau \leq \hat{C}_v R_{jlv}, \tag{2.4}
\]

where

\[
R_{jlv} = \left( \sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m + \alpha_j)^{2\sigma_v}} \right)^{\frac{1}{2}},
\]

with some \( \hat{C}_v > 0 \) and \( \hat{\sigma}_v > \frac{1}{2} \).

Now let \( \varepsilon > 0 \) be arbitrary fixed number,

\[ M_{jv} = M_{jv}(\varepsilon) = C_v R_{jv}\varepsilon^{-1}, \quad j = 1, \ldots, d, \]

and

\[ M_{jlv} = M_{jlv}(\varepsilon) = \hat{C}_v R_{jlv}\varepsilon^{-1}, \quad j = 1, \ldots, r. \]

Then the bounds (2.3) and (2.4) imply

\[
\limsup_{T \to \infty} \mathbb{P} \left( \exists j = 1, \ldots, d: \sup_{s \in K_v} \left| X_{T,n,j}(s) \right| > M_{jv} \right.
\]

or \( \exists (j,l), j = 1, \ldots, r, l = 1, \ldots, l_j: \sup_{s \in K_v} \left| \hat{X}_{T,n,j,l}(s) \right| > M_{jlv} \)

\[
\leq \sum_{j=1}^{d} \limsup_{T \to \infty} \mathbb{P} \left( \sup_{s \in K_j} \left| X_{T,n,j}(s) \right| > M_{jv} \right) + \sum_{j=1}^{r} \sum_{l=1}^{l_j} \limsup_{T \to \infty} \mathbb{P} \left( \sup_{s \in K_j} \left| \hat{X}_{T,n,j,l}(s) \right| > M_{jlv} \right)
\]

\[
\leq \sum_{j=1}^{d} \frac{1}{M_{jv}} \sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K_v} \left| L_n(s + i\tau, \chi_j) \right| d\tau
\]

\[
+ \sum_{j=1}^{r} \sum_{l=1}^{l_j} \frac{1}{M_{jlv}} \sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K_v} \left| \zeta_n(s + i\tau, \alpha_j; a_{jl}) \right| d\tau
\]

\[
< \frac{\varepsilon}{2^{v+1}} + \frac{\varepsilon}{2^{v+1}} = \frac{\varepsilon}{2^{v}}, \quad v \in \mathbb{N}.
\]

This together with (2.2), for all \( n \in \mathbb{N} \), gives

\[
\mathbb{P} \left( \exists j = 1, \ldots, d: \sup_{s \in K_v} \left| X_{n,j}(s) \right| > M_{jv} \text{ or } \exists (j,l), j = 1, \ldots, r, \right.
\]

\[
l = 1, \ldots, l_j: \sup_{s \in K_v} \left| \hat{X}_{n,j,l}(s) \right| > M_{jlv} \right) \leq \frac{\varepsilon}{2^{v}}, \quad v \in \mathbb{N}. \tag{2.5}
\]
Define the set
\[ K_{d,u} = K_{d,u}(\varepsilon) = \left\{ (g_1, \ldots, g_d, \hat{g}_{11}, \ldots, \hat{g}_{1l_1}, \ldots, \hat{g}_{r l_r}) \in H_{d,u}: \right. \]
\[ \sup_{s \in K_v} |g_j(s)| \leq M_{jv}, \quad j = 1, \ldots, r, \]
\[ \sup_{s \in K_v} |\hat{g}_{jl}(s)| \leq M_{jlv}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j, \quad v \in \mathbb{N} \}. \]

Then \( K_{d,u} \) is a compact subset in \( H_{d,u} \). Moreover, in view of (2.5), for all \( n \in \mathbb{N} \),
\[ \mathbb{P}(X_n \in K_{d,u}) > 1 - \varepsilon \sum_{v=1}^{\infty} \frac{1}{2^v} = 1 - \varepsilon, \]
or, by the definition of \( X_n \), we find that \( P_n(K_{d,u}(\varepsilon)) > 1 - \varepsilon \), for all \( n \in \mathbb{N} \).

Thus, we have proved that the family of probability measures \( \{P_n: n \in \mathbb{N}\} \) is tight. Hence, by the Prokhorov theorem, see [2], it is relatively compact. Therefore, there exists a sequence \( \{P_{n_k}\} \subset P_n \) such that \( P_{n_k} \) converges weakly to a certain probability measure \( P \) on \( (H_{d,u}, \mathcal{B}(H_{d,u})) \) as \( k \to \infty \).

In other words,
\[ X_{n_k} \xrightarrow{D} P. \quad (2.6) \]

Now, using (2.1), (2.2), (2.6) and Theorem 4.2 of [2], we obtain that
\[ X_T \xrightarrow{D} P, \quad (2.7) \]
where, \( X_T = X(s) = F(s + i\theta T, \chi, \alpha, \mathfrak{a}) \). Thus, we proved that \( P_T \) converges weakly to \( P \) as \( T \to \infty \).

Similarly to a relation (2.1), we find that, for almost all \( \omega \in \Omega^\kappa \),
\[ \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_{d,u} \left( F(s + i\tau, \chi, \alpha, \omega; \mathfrak{a}), F_n(s + i\tau, \chi, \alpha, \omega; \mathfrak{a}) \right) \, d\tau = 0. \quad (2.8) \]

Let
\[ \hat{X}_{T,n} = \hat{X}(s) = F_n(s + i\theta T, \chi, \alpha, \omega; \mathfrak{a}). \]

Then, by Lemma 2
\[ \hat{X}_{T,n} \xrightarrow{D} P_n. \quad (2.9) \]

Moreover, (2.7) shows that
\[ \hat{X}_n \xrightarrow{D} P. \quad (2.10) \]

Now (2.8)–(2.10) and Theorem 4.2 of [2] show that
\[ \hat{X}_T \xrightarrow{D} P. \]
where
\[ \hat{X}_T = \hat{X}_T(s) = F(s + i\theta T, \chi, \alpha, \omega; \mathfrak{a}). \]

Then, \( \hat{P}_T \) also converges weakly to \( P \) as \( T \to \infty \). The lemma is proved. \( \square \)

Proof of Theorem 2. We apply standard arguments. Let $A$ be a fixed continuity set of the measure $P$ in Lemma 3. Then we have that

$$\lim_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0,T]: F(s + i\tau, \chi, \alpha, \omega; a) \in A\} = P(A). \quad (2.11)$$

We have to prove that $P = P_F$. For this, on the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$, define the random variable

$$\xi(\omega) = \begin{cases} 1, & \text{if } F(s, \chi, \alpha, \omega; a) \in A, \\ 0, & \text{otherwise}. \end{cases}$$

Obviously, the expectation $\mathbb{E}\xi$ of $\xi(\omega)$ is of the form:

$$\mathbb{E}\xi = \int_{\Omega^\kappa} \xi(\omega) \, dm_H^\kappa = m_H^\kappa(\omega \in \Omega^\kappa: F(s, \chi, \alpha, \omega; a) \in A). \quad (2.12)$$

In the sequel, we apply the ergodic theory. We consider the one parameter group $\{\Phi_\tau: \tau \in \mathbb{R}\}$ of measurable measure preserving transformations on $\Omega^\kappa$ given by

$$\Phi_\tau(\omega) = ((p^{-i\tau}; p \in \mathcal{P}), ((m + \alpha_1)^{-i\tau}; m \in \mathbb{N}_0), \ldots, ((m + \alpha_r)^{-i\tau}; m \in \mathbb{N}_0))\omega.$$

In [10, Lemma 7], it is proved that the group $\{\Phi_\tau: \tau \in \mathbb{R}\}$ is ergodic. Hence, the random process $\xi(\Phi_\tau(\omega))$ is ergodic as well. Therefore, the Birkhoff–Khintchine theorem implies the inequality

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \xi(\Phi_\tau(\omega)) \, d\tau = \mathbb{E}\xi. \quad (2.13)$$

On the other hand, the definitions of $\xi$ and $\Phi_\tau$ yield

$$\frac{1}{T} \int_0^T \xi(\Phi_\tau(\omega)) \, d\tau = \frac{1}{T} \text{meas}\{\tau \in [0,T]: F(s + i\tau, \chi, \alpha, \omega; a) \in A\}.$$

This, (2.12) and (2.13) show that

$$\lim_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0,T]: F(s + i\tau, \chi, \alpha, \omega; a) \in A\} = P_F(A).$$

Then in view of (2.11), $P(A) = P_F(A)$. Since $A$ was an arbitrary continuity set of $P$, the later equality holds for all continuity sets of $P$. It is well known that continuity sets constitute a determining class. Consequently, $P(A) = P_F(A)$ for all $A \in \mathcal{B}(H_{d,u})$, and the theorem is proved. $\square$

3 Support

This section is devoted to the explicit form of the support $S_{P_F}$ of the probability measure $P_F$. By the definition, $S_{P_F}$ is a minimal closed subset of $H_{d,u}$ such that $P_F(S_{P_F}) = 1$. Define

$$S = \{g \in H(D): g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$
Theorem 3. Suppose that \( \chi_1, \ldots, \chi_d \) are pairwise non-equivalent Dirichlet characters, and parameters \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \). Then the support of \( P_F \) is the set \( S^d \times H^u(D) \).

Proof. The set \( H_{d,u} \) is separable, therefore \[ \mathcal{B}(H_{d,u}) = \mathcal{B}(H^d(D)) \times \mathcal{B}(H^u(D)). \]
Therefore, it suffices to consider \( P_F(A \times B) \), where \( A = \mathcal{B}(H^d(D)) \) and \( B = \mathcal{B}(H^u(D)) \). Let
\[
L(s, \chi, \omega) = (L(s, \chi_1, \omega), \ldots, L(s, \chi_d, \omega)),
\]
\[
\zeta(s, \alpha, \omega; a) = (\zeta(s, \alpha_1, \omega_1; a_1), \ldots, \zeta(s, \alpha_1, \omega_1; a_{1l_1}), \ldots),
\]
where \( \omega = (\omega_1, \ldots, \omega_r) \). Since the measure \( m_H^r \) is a product of the measures \( m_H \) and \( m_H^r \), we have that
\[
P_F(A \times B) = m_H^r(\omega \in \Omega^r : F(s, \chi, \omega; a) \in A \times B)
\]
\[
m_H(\omega \in \Omega : L(s, \chi, \omega) \in A) \times m_H^r(\omega \in \Omega^r : \zeta(s, \alpha, \omega; a) \in B).
\]
In [11], it was obtained that \( S^d \) is a minimal closed set such
\[
m_H(\omega \in \Omega : L(s, \chi, \omega) \in S^d) = 1,
\]
and in [8], it was proved that \( H^u(D) \) is a minimal closed set such that
\[
m_H^r(\omega \in \Omega^r : \zeta(s, \alpha, \omega; a) \in H^u(D)) = 1.
\]
These equalities together with (3.1) prove the theorem. \( \square \)

4 Proof of Theorem 1

We will apply the Mergelyan theorem on approximation of analytic functions by polynomials [18] which asserts that if \( K \subset \mathbb{C} \) is a compact subset with connected complement, and \( g(s) \) is a function continuous on \( K \) and analytic in the interior of \( K \), then, for every \( \varepsilon > 0 \), there is a polynomial \( p(s) \) such that
\[
\sup_{s \in K} |g(s) - p(s)| < \varepsilon.
\]

Proof of Theorem 1. By the Mergelyan theorem, there exist polynomials \( p_j(s), j = 1, \ldots, d \), and \( p_{jl}(s), j = 1, \ldots, r, l = 1, \ldots, l_j \), such that
\[
\sup_{1 \leq j \leq d} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{4} \quad (4.1)
\]
and
\[
\sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |f_{jl}(s) - p_{jl}(s)| < \frac{\varepsilon}{2} \quad (4.2)
\]

If $\epsilon$ is sufficiently small, we have that $p_j(s) \neq 0$ on $K_j$, $j = 1, \ldots, d$. Therefore, there exists a continuous branch $\log p_j(s)$ which is analytic in the interior of $K_j$, $j = 1, \ldots, d$. Applying the Mergelyan theorem once more, we can find polynomials $q_j(s)$, $j = 1, \ldots, d$, such that

$$\sup_{1 \leq j \leq d} \sup_{s \in K_j} |p_j(s) - e^{q_j(s)}| < \frac{\epsilon}{4}. $$

Combining this with (4.1) gives

$$\sup_{1 \leq j \leq d} \sup_{s \in K_j} |f_j(s) - e^{q_j(s)}| < \frac{\epsilon}{2}. \tag{4.3}$$

By Theorem 3,

$$\left(e^{q_1(s)}, \ldots, e^{q_d(s)}, p_{11}(s), \ldots, p_{1l_1}(s), \ldots, p_{r1}(s), \ldots, p_{rl}(s)\right) \in S_{P_F}.$$ 

Therefore, setting

$$G = \left\{ g \in H_{d,r} : \sup_{1 \leq j \leq d} \sup_{s \in K_j} |g_j(s) - e^{q_j(s)}| < \frac{\epsilon}{2}, \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{j,l}} |g_{jl}(s) - p_{jl}(s)| < \frac{\epsilon}{2} \right\},$$

we obtain, by Theorem 2, that

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{ \tau \in [0,T] : F(s + i\tau, \chi, a; \alpha) \in G \} \geq p_{F}(G) > 0.$$ 

This, definition of $G$ and (4.2) and (4.3) complete the proof of the theorem. $\square$

References


