Mathematical Modelling and Analysis
Volume 16 Number 2, June 2011, 199–219
Doi:10.3846/13926292.2011.578282

www.tandf.co.uk/journals/TMMA
Publisher: Taylor&Francis and VGTU
Online ISSN: 1648-3510
Print ISSN: 1392-6292

Reconstruction of a Source Term in a Parabolic Integro-Differential Equation from Final Data*

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Received November 30, 2010; revised March 31, 2011; published online May 1, 2011

Abstract. The identification of a source term in a parabolic integro-differential equation is considered. We study the existence of the quasi-solution to this problem, Tikhonov regularization and a related gradient method.

Keywords: Inverse problem, integro-differential equation, quasi-solution.

AMS Subject Classification: 35R30; 80A23.

1 Introduction

Heat flow processes in media with memory are governed by parabolic integro-differential equations [7]. A number of papers is devoted to inverse problems to determine kernels of these equations in different formulations making use of measurements over time (see e.g. [4, 6, 7, 8, 11, 13, 14]).

Recently some papers appeared that deal with the reconstruction of source terms or coefficients of these equations making use of final or integral over-determination [5, 12]. In particular, the authors' paper [5] extends former existence and uniqueness results of Isakov [3] to the integro-differential case. The existence of the solutions to the inverse problems to determine unknown source terms from final over-determination of the temperature requires sufficient regularity and a certain monotonicity of a time-component of this term.

In the present paper we follow another approach. Instead of the conventional solution, we deal with the quasi-solution of the inverse problem that uses final data. Then we can build up a theory without any smoothness or monotonicity restrictions on the source. Similar results in the case of the parabolic differential equation without an integral term in the one-dimensional case were obtained by Hasanov [2]. Quasi-solutions of other integro-differential inverse problems were studied in [1, 9].

* Supported by Estonian Science Foundation grant 7728 and Estonian Ministry of Education and Science TF themes SF0140011s09, SF0140083s08.

2 Direct Problem

Let Ω be a n-dimensional domain with sufficiently smooth boundary Γ and $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\operatorname{meas} \Gamma_1 \cap \Gamma_2 = 0$. Assume that for any $j \in \{1;2\}$ it holds either $\Gamma_j = \emptyset$ or $\operatorname{meas} \Gamma_j > 0$. Denote $\Omega_T = \Omega \times (0,T)$, $\Gamma_{1,T} = \Gamma_1 \times (0,T)$, $\Gamma_{2,T} = \Gamma_2 \times (0,T)$. Consider the problem (direct problem) to find $u(x,t) : \Omega_T \to \mathbb{R}$ such that

$$u_t = Au - m * Au + f + \nabla\phi \quad \text{in } \Omega_T, \tag{2.1}$$

$$u = u_0 \quad \text{in } \Omega \times \{0\}, \tag{2.2}$$

$$u = g \quad \text{in } \Gamma_{1,T}, \tag{2.3}$$

$$-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u = \vartheta u + h \quad \text{in } \Gamma_{2,T} \tag{2.4}$$

where

$$Av = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}v\right) + av,$$

$$\nu_A = \sum_{j=1}^{n} a_{ij}\nu_j, \quad \nu = (\nu_1, \ldots, \nu_n) \text{ - outer normal of } \Gamma_2,$$

$a_{ij}, a, u_0 : \Omega \to \mathbb{R}$, $f : \Omega_T \to \mathbb{R}$, $\phi : \Omega_T \to \mathbb{R}^n$, $g : \Gamma_{1,T} \to \mathbb{R}$, $\vartheta : \Gamma_2 \to \mathbb{R}$, $h : \Gamma_{2,T} \to \mathbb{R}$, $m : (0,T) \to \mathbb{R}$ are given functions and

$$m * w(t) = \int_0^t m(t-\tau)w(\tau)\,d\tau$$

denotes the time convolution. In case $\Gamma_1 = \emptyset$ ($\Gamma_2 = \emptyset$), the boundary condition (2.3) ((2.4)) is dropped.

The problem (2.1)–(2.4) describes the heat flow in a body Ω with the thermal memory. Concerning the physical background we refer the reader to [7]. The solution u is the temperature of the body and m is the heat flux relaxation (or memory) kernel. The boundary condition (2.4) is of the third kind where the term $-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u$ equals the heat flux in the direction of the co-normal vector.

Let us introduce some additional notation. Let X be a Banach space. We denote by $C([0,T];X)$ the space of abstract continuous functions from $[0,T]$ to X endowed with the usual maximum norm $\|v\|_{C([0,T];X)} := \max_{t\in[0,T]} \|v(x)\|$. Moreover, let

$$L^2((0,T);X) := \left\{ v : (0,T) \to X \colon \|v\|_{L^2((0,T);X)} = \left[\int_0^T \|v(t)\|^2\,dt \right]^{1/2} < \infty \right\}.$$

In addition, we need spaces of fractional order and anisotropic spaces. To this end, let us first introduce the following notation for difference quotients of x- and (x,t)-dependent functions with powers:

$$\langle v\rangle_p(x_1,x_2) := \tfrac{v(x_1)-v(x_2)}{|x_1-x_2|^p}, \qquad \langle v\rangle_p(x_1,x_2;t) := \tfrac{v(x_1,t)-v(x_2,t)}{|x_1-x_2|^p},$$

$$\langle v\rangle_p(x;t_1,t_2) := \tfrac{v(x,t_1)-v(x,t_2)}{|t_1-t_2|^p},$$

where $|x|$ denotes the Euclidean norm of x in the space $\mathbb{R}^n$. For any $l \geq 0$ we introduce the Sobolev–Slobodeckij spaces (cf. [10, 15])

$$W_2^l(\Omega) = \Big\{v\colon \|v\|_{W_2^l(\Omega)} := \sum_{|\alpha|\leq[l]} \Big[\int_\Omega |D_x^\alpha v(x)|^2\,dx\Big]^{\frac{1}{2}} + \Theta_l \sum_{|\alpha|=[l]} \Big[\int_\Omega dx_1 \int_\Omega \big|\langle D_x^\alpha v\rangle_{\frac{n}{2}+l-[l]}(x_1,x_2)\big|^2\,dx_2\Big]^{\frac{1}{2}} < \infty\Big\},$$

$$W_2^{l,\frac{l}{2}}(\Omega_T) = \Big\{v\colon \|v\|_{W_2^{l,\frac{l}{2}}(\Omega_T)} := \sum_{2j+|\alpha|\leq[l]} \Big[\int_0^T \int_\Omega |D_t^j D_x^\alpha v(x,t)|^2\,dx\,dt\Big]^{\frac{1}{2}} + \Theta_l \sum_{2j+|\alpha|=[l]} \Big[\int_0^T dt \int_\Omega dx_1 \int_\Omega \big|\langle D_t^j D_x^\alpha v\rangle_{\frac{n}{2}+l-[l]}(x_1,x_2;t)\big|^2\,dx_2\Big]^{\frac{1}{2}} + \Theta_{\frac{l}{2}} \sum_{\substack{l-2j-|\alpha|\\ \in(0,2)}} \Big[\int_\Omega dx \int_0^T dt_1 \int_0^T \big|\langle D_t^j D_x^\alpha v\rangle_{\frac{1}{2}+\frac{l-2j-|\alpha|}{2}}(x;t_1,t_2)\big|^2\,dt_2\Big]^{\frac{1}{2}} < \infty\Big\}.$$

Here $\alpha = (\alpha_1,\ldots,\alpha_n)$ with $\alpha_i \in \{0,1,2,\ldots\}$ is the multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_x^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ and $D_t^j v = \frac{\partial^j v}{\partial t^j}$. Moreover, $[l]$ is the greatest integer $\leq l$ and $\Theta_l = 0$ and $\Theta_l = 1$ in the cases of integer l and non-integer l, respectively. The definition of $W_2^{l,\frac{l}{2}}$ is in a standard manner extended from Ω_T to the boundary components $\Gamma_{1,T}$ and $\Gamma_{2,T}$ (for details see [10]).

Now we return to the direct problem (2.1)–(2.4). Throughout the paper we assume the following basic regularity conditions on the coefficients, the kernel and the initial and boundary functions:

$$a_{ij} \in C^1(\overline{\Omega}), \quad a_{ij} = a_{ji}, \quad a \in C(\overline{\Omega}), \quad \vartheta \in C(\overline{\Gamma}_2), \quad \vartheta \geq 0, \tag{2.5}$$

$$m \in L^1(0,T), \quad g \in W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{1,T}), \quad h \in L^2(\Gamma_{2,T}), \tag{2.6}$$

$$u_0 \in L^2(\Omega), \quad f \in L^2(\Omega_T), \quad \phi = (\phi_1,\ldots,\phi_n) \in (L^2(\Omega_T))^n \tag{2.7}$$

and the ellipticity condition

$$\sum_{i,j=1}^n a_{ij}\lambda_i\lambda_i \geq \epsilon|\lambda|^2, \quad x \in \overline{\Omega},\ \lambda \in \mathbb{R}^n \text{ with some } \epsilon > 0. \tag{2.8}$$

The first aim is to reformulate the problem (2.1)–(2.4) in a weak form. Let us suppose that (2.1)–(2.4) has a classical solution $u \in W_2^{2,1}(\Omega_T)$ and the term ϕ satisfies the following additional conditions: $\frac{\partial}{\partial x_i}\phi_i \in (L^2(\Omega_T))^n$, $i = 1,\ldots,n$, $\phi|_{\Gamma_{2,T}} = 0$. Then, we multiply (2.1) with a test function η from the space

$$\mathcal{T}(\Omega_T) = \Big\{\eta \in L^2((0,T);W_2^1(\Omega))\colon \eta_t \in L^2((0,T);L^2(\Omega)),\ \eta|_{\Gamma_1} = 0 \text{ in case } \Gamma_1 \neq \emptyset\Big\}$$

and integrate by parts with respect to time and space variables. We obtain the following relation:

$$
\begin{aligned}
0 = &\int_{\Omega} \left[u(x,T)\eta(x,T) - u_0(x)\eta(x,0)\right]\, dx - \iint_{\Omega_T} u\eta_t \, dx\, dt \\
&+ \iint_{\Omega_T} \Bigg[\sum_{i,j=1}^{n} a_{ij}(u_{x_j} - m * u_{x_j})\eta_{x_i} - a(u - m*u)\eta\Bigg] dx\, dt \\
&+ \iint_{\Gamma_{2,T}} (\vartheta u + h)\eta \, d\Gamma\, dt - \iint_{\Omega_T} (f\eta - \phi\cdot\nabla\eta)\, dx\, dt. \qquad (2.9)
\end{aligned}
$$

This relation makes sense also in a more general case when ϕ satisfies only (2.7) and u doesn't have regular first order time and second order spatial derivatives. We call a *weak solution* of the problem (2.1)–(2.4) a function from the space

$$\mathcal{U}(\Omega_T) = C([0,T]; L^2(\Omega)) \cap L^2((0,T); W_2^1(\Omega))$$

that satisfies the relation (2.9) for any $\eta \in \mathcal{T}(\Omega_T)$ and in case $\Gamma_1 \neq \emptyset$ fulfills the boundary condition (2.3).

Theorem 1. *The problem* (2.1)–(2.4) *has a unique weak solution. If, in addition,* $\phi = 0$, $g \in W_2^{\frac{3}{2},\frac{3}{4}}(\Gamma_{1,T})$, $h \in W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{2,T})$, $u_0 \in H^1(\Omega)$ *and* $u_0 = g$ *on* $\Gamma_1 \times \{0\}$ *then this solution belongs to the space* $W_2^{2,1}(\Omega_T)$ *and satisfies* (2.1)–(2.4) *in the classical sense.*

Proof. It is well known (see e.g. [10]) that in the particular case $m = 0$ the solution exists, is unique and the operator $\mathcal{H}$, that assigns to the data vector u_0, g, h, f, ϕ the weak solution is Lipschitz-continuous from the space $L^2(\Omega)\times W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{1,T})\times L^2(\Gamma_{2,T})\times L^2(\Omega_T)^{n+1}$ to the space $\mathcal{U}(\Omega_T)$. Let us denote $\mathcal{G}(f,\phi) = \mathcal{H}(0,0,0,f,\phi)$. Then, denoting by $\widehat{u}$ the solution corresponding to $m = 0$, the problem (2.1)–(2.4) for u is in $\mathcal{U}(\Omega_T)$ equivalent to the following operator equation for the function $v = u - \widehat{u}$:

$$v = \mathcal{F}\widehat{u} + \mathcal{F}v \qquad (2.10)$$

with the linear operator $\mathcal{F}v = \mathcal{G}(-m*(av), -m*(\sum_{j=1}^{n} a_{ij}v_{x_j}))$. We are going to estimate $\mathcal{F}$. To this end, we make use of the following inequality that immediately follows from the estimate (19) in [5]:

$$\|m*w\|_{L^2(\Omega_t)} \le \int_0^t |m(t-\tau)|\|w\|_{L^2(\Omega_\tau)}\, d\tau, \quad t\in(0,T). \qquad (2.11)$$

Here $\Omega_t = \Omega\times(0,t)$ for $t\in(0,T)$ and w is an arbitrary element of $L^2(\Omega_T)$. Moreover, we define the cutting operator P_t by the formula

$$P_t w = \begin{cases} w & \text{in } \Omega_t, \\ 0 & \text{in } \Omega_T \setminus \Omega_t. \end{cases}$$

Note that it holds $\mathcal{G}(P_t f, P_t\phi)(x,t) = \mathcal{G}(f,\phi)(x,t)$ for any $(x,t) \in \Omega_t$. Therefore, observing the Lipschitz-continuity of $\mathcal{G}$ and (2.11) we can estimate as follows:

$$\begin{aligned}
\|\mathcal{F}v\|_{\mathcal{U}(\Omega_t)} &= \left\| \mathcal{G}\Big(-m*(av), -m*\Big(\sum_{j=1}^n a_{ij}v_{x_j}\Big)\Big)\right\|_{\mathcal{U}(\Omega_t)} \\
&= \left\| \mathcal{G}\Big(-P_t\left[m*(av)\right], -P_t\Big[m*\Big(\sum_{j=1}^n a_{ij}v_{x_j}\Big)\Big]\Big)\right\|_{\mathcal{U}(\Omega_t)} \\
&\le \left\| \mathcal{G}\Big(-P_t\left[m*(av)\right], -P_t\Big[m*\Big(\sum_{j=1}^n a_{ij}v_{x_j}\Big)\Big]\Big)\right\|_{\mathcal{U}(\Omega_T)} \\
&\le C_1\Big[\|P_t\left[m*(av)\right]\|_{L^2(\Omega_T)} + \sum_{i=1}^n \Big\|P_t\Big[m*\sum_{j=1}^n a_{ij}v_{x_j}\Big]\Big\|_{L^2(\Omega_T)}\Big] \\
&= C_1\Big[\|m*(av)\|_{L^2(\Omega_t)} + \sum_{i=1}^n \Big\|m*\sum_{j=1}^n a_{ij}v_{x_j}\Big\|_{L^2(\Omega_t)}\Big] \\
&\le C_2\int_0^t |m(t-\tau)|(\|v\|_{L^2(\Omega_\tau)} + \|\nabla v\|_{L^2(\Omega_\tau)})\,d\tau \\
&\le C_2\int_0^t |m(t-\tau)|\|v\|_{\mathcal{U}(\Omega_\tau)}\,d\tau
\end{aligned}$$

for any $t \in (0,T)$ with some constants C_1, C_2. Now we introduce the weighted norms in $\mathcal{U}(\Omega_T)$: $\|v\|_\sigma = \sup_{0<t<T} e^{-\sigma t}\|v\|_{\mathcal{U}(\Omega_t)}$ where $\sigma \ge 0$. Using the deduced estimate for $\mathcal{F}$ we obtain

$$\begin{aligned}
\|\mathcal{F}v\|_\sigma &\le C_2 \sup_{0<t<T} e^{-\sigma t}\int_0^t |m(t-\tau)|\|v\|_{\mathcal{U}(\Omega_\tau)}\,d\tau \\
&= C_2 \sup_{0<t<T}\int_0^t e^{-\sigma(t-\tau)}|m(t-\tau)|e^{-\sigma\tau}\|v\|_{\mathcal{U}(\Omega_\tau)}\,d\tau \\
&\le C_2\int_0^T e^{-\sigma s}|m(s)|\,ds\,\|v\|_\sigma.
\end{aligned}$$

Since $\int_0^T e^{-\sigma s}|m(s)|\,ds \to 0$ as $\sigma \to \infty$, the operator $\mathcal{F}$ is a contraction for sufficiently large σ. Consequently, (2.10) has a unique solution in $\mathcal{U}(\Omega_T)$. This proves the existence of the unique weak solution of (2.1)–(2.4).

Secondly, let us prove the classical solvability assertion of the theorem. Again, we use the results in case $m = 0$. It is known [15] that in case $m = 0$ the solution belongs to $W_2^1(\Omega_T)$ and the operator $\mathcal{H}^1$ that assigns to the data vector u_0, g, h, f the classical solution is Lipschitz-continuous from the space $H^1(\Omega) \times W_2^{\frac{3}{2},\frac{3}{4}}(\Gamma_{1,T}) \times W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{2,T}) \times L^2(\Omega_T)$ to the space $W_2^{2,1}(\Omega_T)$. Define $\mathcal{G}^1(h,f) = \mathcal{H}^1(0,0,h,f)$. The problem for u is equivalent to the following operator equation for $v = u - \widehat{u}$:

$$v = \mathcal{F}^1\widehat{u} + \mathcal{F}^1 v, \tag{2.12}$$

where $\mathcal{F}^1 v = \mathcal{G}^1(-m * \nu_A \cdot \nabla v|_{\Gamma_{2,T}}, -m * Av)$. This time we have to introduce a more complicated extension operator instead of P_t because the argument of $\mathcal{F}^1$ has traces on slices $\Omega \times \{t\}$. Let us define

$$\widetilde{P}_t w(x,s) = \begin{cases} w(x,s) & \text{for } s < t, \\ w(x, 2t - s) & \text{for } t < s < \min\{2t; T\}, \\ 0 & \text{for } s > 2t \text{ in case } 2t < T. \end{cases}$$

Then, since the function v in the range of $\mathcal{F}^1$ satisfies $v|_{t=0} = 0$, it holds $\widetilde{P}_t v \in W_2^{2,1}(\Omega_T)$ for $t \in (0,T)$. Moreover, $\mathcal{G}^1(\widetilde{P}_t \widetilde{h}, \widetilde{P}_t \widetilde{f})(x,t) = \mathcal{G}^1(\widetilde{h}, \widetilde{f})(x,t)$ for any $(x,t) \in \Omega_t$ and $\|\widetilde{P}_t \widetilde{h}\|_{W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{2,T})} \le 2\|\widetilde{h}\|_{W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{2,t})}$, $\|\widetilde{P}_t \widetilde{f}\|_{L^2(\Omega_T)} \le 2\|\widetilde{f}\|_{L^2(\Omega_t)}$, where $\widetilde{h} = m * \nu_A \cdot \nabla v|_{\Gamma_{2,T}}$ and $\widetilde{f} = m * Av$. Consequently, in view of the Lipschitz-continuity of $\mathcal{G}^1$ we deduce

$$\begin{aligned} \|\mathcal{F}^1 v\|_{W_2^{2,1}(\Omega_t)} &= \left\|\mathcal{G}^1\left(-m * \nu_A \cdot \nabla v|_{\Gamma_{2,T}}, -m * Av\right)\right\|_{W_2^{2,1}(\Omega_t)} \\ &= \left\|\mathcal{G}^1\left(-P_t\left[m * \nu_A \cdot \nabla v|_{\Gamma_{2,T}}\right], -P_t\left[m * Av\right]\right)\right\|_{W_2^{2,1}(\Omega_t)} \\ &\le \left\|\mathcal{G}^1\left(-P_t\left[m * \nu_A \cdot \nabla v|_{\Gamma_{2,T}}\right], -P_t\left[m * Av\right]\right)\right\|_{W_2^{2,1}(\Omega_T)} \\ &\le C_3\left[\|P_t\left[m * \nu_A \cdot \nabla v\right]\|_{W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{2,T})} + \|P_t\left[m * Av\right]\|_{L^2(\Omega_T)}\right] \\ &\le 2C_3\left[\|m * \nu_A \cdot \nabla v\|_{W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{2,t})} + \|m * Av\|_{L^2(\Omega_t)}\right] \end{aligned} \tag{2.13}$$

for any $t \in (0,T)$ with some constant C_3 and $\Gamma_{2,t} = \Gamma_2 \times (0,t)$. Using the trace theorem for Sobolev–Slobodeckij spaces [10] and the relation $(m * v)_t = m * v_t$, that holds due to $v|_{t=0} = 0$, we compute

$$\begin{aligned} \|m * \nu_A \cdot \nabla v\|_{W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{2,t})} &= \|\nu_A \cdot \nabla(m * v)\|_{W_2^{\frac{1}{2},\frac{1}{4}}(\Gamma_{2,t})} \le C_4\|m * v\|_{W_2^{2,1}(\Omega_t)} \\ &= C_4\left[\sum_{|\alpha| \le 2} \|m * D_x^\alpha v\|_{L^2(\Omega_t)} + \|m * v_t\|_{L^2(\Omega_t)}\right] \end{aligned}$$

with some constant C_4. Applying this estimate in (2.13) and using (2.11) we deduce

$$\|\mathcal{F}^1 v\|_{W_2^{2,1}(\Omega_t)} \le C_5 \int_0^t |m(t-\tau)|\|v\|_{W_2^{2,1}(\Omega_\tau)}\, d\tau, \quad t \in (0,T)$$

with a constant C_5. We define the weighted norms

$$\|v\|_\sigma^* = \sup_{0<t<T} e^{-\sigma t}\|v\|_{W_2^{2,1}(\Omega_t)}$$

in the space $W_2^{2,1}(\Omega_T)$ and, as in the first part of the proof, show that $\mathcal{F}^1$ is a contraction in $W_2^{2,1}(\Omega_T)$ if σ is sufficiently large. This proves the unique solvability of (2.12) and in turn the classical solvability assertion of theorem. □

3 Formulation of Inverse Problem. Existence of Quasi-Solution

Let $\widehat{\mathcal{F}}$ be a linear closed subspace of $L^2(\Omega_T)$. Suppose that the source term f is of the following form: $f = f_0 + F$, where $f_0 \in L^2(\Omega_T)$ is known. We pose an inverse problem to determine the function $F \in \widehat{\mathcal{F}}$ making use of the final measurement

$$u(x,T) = u_T(x), \quad x \in \Omega.$$

More precisely, we will search a *quasi-solution* of this problem. This is a solution of the following minimization problem for the cost functional: find

$$F^* = \arg\min_{F\in\mathcal{F}} J(F), \quad J(F) = \|u(\cdot,T;F) - u_T\|^2_{L^2(\Omega)}, \tag{3.1}$$

where $\mathcal{F} \subseteq \widehat{\mathcal{F}}$ is a subset including constraints. Here $u(x,t;F)$ stands for the solution of the direct problem corresponding to the given F.

Let us introduce some cases of $\widehat{\mathcal{F}}$.

Case 1. Define $\widehat{\mathcal{F}} = \{F\colon F(x,t) = \varkappa(t)w(x),\ w \in L^2(\Omega)\}$, where $\varkappa \in L^2(0,T)$, $\varkappa \neq 0$ is a prescribed function.

Case 2. Let Ω be a cylinder: $\Omega = S\times(0,l)$, where for any $x = (x_1,\dots,x_n) \in \Omega$ we have $\overline{x} = (x_1,\dots,x_{n-1}) \in S$, $x_n \in (0,l)$. Define $\widehat{\mathcal{F}} = \{F\colon F(x,t) = \varkappa(x_n)w(\overline{x},t),\ w \in L^2(S_T)\}$, where $\varkappa \in L^2(0,l)$, $\varkappa \neq 0$ is a prescribed function and $S_T = S \times (0,T)$.

Case 3. Define $\widehat{\mathcal{F}} = \{F\colon F(x,t) = \sum_{j=1}^N w_j\varkappa_j(x,t),\ w = (w_j)_{j=1,\dots,N} \in \mathbb{R}^N\}$, where $\varkappa = (\varkappa_j)_{j=1,\dots,N} \in (L^2(\Omega_T))^N$, $\varkappa \neq 0$ is a prescribed vector-function. In practice, the component $\varkappa_j$ may be the characteristic function of a subdomain $\Omega_j \subset \Omega$.

Now let us consider the first variation of the cost functional

$$\begin{aligned}\Delta J(F) &= J(F+\Delta F) - J(F)\\ &= 2\int_\Omega [u(x,T;F) - u_T(x)]\Delta u(x,T;F)\,dx + \int_\Omega [\Delta u(x,T;F)]^2\,dx, \end{aligned}\tag{3.2}$$

where $\Delta u(x,t;F) = u(x,t;F+\Delta F) - u(x,t;F)$. By Theorem 1, the function Δu belongs to $W_2^{2,1}(\Omega_T)$ and solves the following problem in the classical sense:

$$\Delta u_t = A\Delta u - m * A\Delta u + \Delta F \quad \text{in } \Omega_T, \tag{3.3}$$
$$\Delta u = 0 \quad \text{in } \Omega\times\{0\}, \tag{3.4}$$
$$\Delta u = 0 \quad \text{in } \Gamma_{1,T}, \tag{3.5}$$
$$-\nu_A\cdot\nabla\Delta u + m * \nu_A\cdot\nabla\Delta u = \vartheta\Delta u \quad \text{in } \Gamma_{2,T}. \tag{3.6}$$

Moreover, let us introduce the following adjoint problem with the solution $\psi(x,t;F)$:

$$\psi_t(x,t;F) = -A\psi(x,t;F) + \int_t^T m(\tau-t)A\psi(x,\tau;F)\,d\tau \quad \text{in } \Omega_T, \tag{3.7}$$
$$\psi(x,T;F) = 2[u(x,T;F) - u_T(x)] \quad \text{in } \Omega, \tag{3.8}$$

$$\psi(x,t;F)=0 \quad \text{in } \Gamma_{1,T}, \tag{3.9}$$

$$-\nu_A\cdot\nabla\psi(x,t;F)+\int_t^T m(\tau-t)\nu_A\cdot\nabla\psi(x,\tau;F)\,d\tau=\vartheta\psi(x,t;F) \quad \text{in } \Gamma_{2,T}. \tag{3.10}$$

It is easy to see that the equivalent problem for $\tilde{u}(x,t)=\psi(x,T-t;F)$ is of the form (2.1)–(2.4) with homogeneous differential equation and boundary conditions and the initial condition $\tilde{u}=2[u(\cdot,T;F)-u_T]\in L^2(\Omega)$ in $\Omega\times\{0\}$. Therefore, applying Theorem 1 we conclude that problem (3.7)–(3.10) has a unique weak solution. The weak problem for $\psi(x,T-t;F)$ reads

$$\begin{aligned}
0=&\int_\Omega[\psi(x,0;F)\eta(x,T)-2[u(x,T;F)-u_T(x)]\eta(x,0)]\,dx\\
&-\iint_{\Omega_T}\psi(x,T-t;F)\eta_t(x,t)\,dx\,dt+\iint_{\Omega_T}\Big[\sum_{i,j=1}^n a_{ij}(x)\Big(\psi_{x_j}(x,T-t;F)\\
&-\int_0^t m(t-\tau)\psi_{x_j}(x,T-\tau;F)\,d\tau\Big)\eta_{x_i}(x,t)\\
&-a(x)\Big(\psi(x,T-t;F)-\int_0^t m(t-\tau)\psi(x,T-\tau;F)\,d\tau\Big)\eta(x,t)\Big]\,dx\,dt\\
&+\iint_{\Gamma_{2,T}}\vartheta\psi(x,T-t;F)\eta(x,t)\,d\Gamma\,dt \quad \forall\eta\in\mathcal{T}(\Omega_T).
\end{aligned} \tag{3.11}$$

Lemma 1. *It holds the following formula:*

$$2\int_\Omega[u(x,T;F)-u_T(x)]\Delta u(x,T,F)\,dx=\iint_{\Omega_T}\psi(x,t;F)\Delta F(x,t)\,dx\,dt. \tag{3.12}$$

Proof. Since $\Delta u\in W_2^{2,1}(\Omega_T)$ satisfies the homogeneous boundary condition on Γ_1, it holds $\Delta u(x,T-t,F)\in\mathcal{T}(\Omega_T)$. Let us use the test function $\eta(x,t)=\Delta u(x,T-t,F)$ in (3.11). This yields (changing the variable t by $T-t$ under the integrals and observing that $\eta(x,T)=0$ and omitting F in the arguments for the sake of shortness)

$$\begin{aligned}
0=&-2\int_\Omega[u(x,T)-u_T(x)]\Delta u(x,T)]\,dx+\iint_{\Omega_T}\psi(x,t)\Delta u_t(x,t)\,dx\,dt\\
&+\iint_{\Omega_T}\Big[\sum_{i,j=1}^n a_{ij}\Big(\psi_{x_j}(x,t)-\int_0^t m(t-\tau)\psi_{x_j}(x,\tau)\,d\tau\Big)\Delta u_{x_j}(x,t)\\
&-a(x)\Big(\psi(x,t)-\int_0^t m(t-\tau)\psi(x,\tau)\,d\tau\Big)\Delta u(x,t)\Big]\,dx\,dt\\
&+\iint_{\Gamma_{2,T}}\vartheta\psi(x,t)\Delta u(x,t)\,d\Gamma\,dt.
\end{aligned} \tag{3.13}$$

On the other hand, the problem (3.3)–(3.6) in the weak form reads

$$\begin{aligned} 0 = &\int\limits_{\Omega} \Delta u(x,T)\zeta(x,T)\,dx - \iint\limits_{\Omega_T} \Delta u \zeta_t \,dx\,dt \\ &+ \iint\limits_{\Omega_T} \Big[\sum_{i,j=1}^{n} a_{ij}(\Delta u_{x_j} - m * \Delta u_{x_j})\zeta_{x_i} - a(\Delta u - m * \Delta u)\zeta \Big] dx\,dt \\ &+ \iint\limits_{\Gamma_{2,T}} \vartheta \Delta u \zeta \,d\Gamma\,dt - \iint\limits_{\Omega_T} \Delta F \zeta \,dx\,dt \quad \forall \zeta \in \mathcal{T}(\Omega_T). \end{aligned} \tag{3.14}$$

Since $\Delta u \in W_2^{2,1}(\Omega_T)$ has the regular time derivative, we can integrate by parts the integral $\iint\limits_{\Omega_T} \Delta u \zeta_t \,dx\,dt$ in (3.14). This results in the relation

$$\begin{aligned} 0 = &\iint\limits_{\Omega_T} \Delta u_t \zeta \,dx\,dt + \iint\limits_{\Omega_T} \Big[\sum_{i,j=1}^{n} a_{ij}(\Delta u_{x_j} - m * \Delta u_{x_j})\zeta_{x_i} \\ &- a(\Delta u - m * \Delta u)\zeta \Big] dx\,dt + \iint\limits_{\Gamma_{2,T}} \vartheta \Delta u \zeta \,d\Gamma\,dt - \iint\limits_{\Omega_T} \Delta F \zeta \,dx\,dt. \end{aligned} \tag{3.15}$$

It is important that this relation doesn't contain the time derivative of the test function ζ. Therefore, we can extend the set of test functions of (3.15) from $\mathcal{T}(\Omega_T)$ to $\mathcal{U}_0(\Omega_T) = \{\zeta \in \mathcal{U}(\Omega_T)\colon \zeta|_{\Gamma_{1,T}} = 0 \text{ in case } \Gamma_2 \neq \emptyset\}$. In particular, it is possible to take the test function $\zeta = \psi \in \mathcal{U}_0(\Omega_T)$. Then we obtain

$$\begin{aligned} 0 = &\iint\limits_{\Omega_T} \Delta u_t \psi \,dx\,dt + \iint\limits_{\Omega_T} \Big[\sum_{i,j=1}^{n} a_{ij}(\Delta u_{x_j} - m * \Delta u_{x_j})\psi_{x_i} \\ &- a(\Delta u - m * \Delta u)\psi \Big] dx\,dt + \iint\limits_{\Gamma_{2,T}} \vartheta \Delta u \psi \,d\Gamma\,dt - \iint\limits_{\Omega_T} \Delta F \psi \,dx\,dt. \end{aligned} \tag{3.16}$$

Subtracting (3.16) from (3.13) and changing the order of integration in convolution terms we deduce the formula (3.12). Lemma is proved. □

Theorem 2. *Let $\mathcal{F}$ be a bounded, closed and convex subset of $\widehat{\mathcal{F}}$. Then the problem* (3.1) *has a solution in $\mathcal{F}$. Moreover, the set of all solutions $\mathcal{F}^*$ form a closed convex subset of $\mathcal{F}$.*

Proof. The assertion follows from Weierstrass existence theorem (see [16, Section 2.5]) once we have proved that $J(F)$ is weakly sequentially lower semicontinuous in $\mathcal{F}$, i.e.

$$J(F) \le \liminf_{n\to\infty} J(F_n) \quad \text{as } F_n \rightharpoonup F \text{ in } \mathcal{F} \tag{3.17}$$

and convex, i.e.

$$J(\gamma F_1 + (1-\gamma)F_2) \le \gamma J(F_1) + (1-\gamma)J(F_2) \quad \forall \gamma \in [0,1],\ F_1, F_2 \in \mathcal{F}.$$

Let us compute:

$$\begin{aligned} J(F) &= \int_\Omega [u(x,T;F) - u_T(x)]^2\,dx = \int_\Omega [u(x,T;F_n) - u_T(x)]^2\,dx \\ &\quad - \int_\Omega [u(x,T;F_n) - u(x,T;F)]^2\,dx \\ &\quad - 2\int_\Omega [u(x,T;F) - u_T(x)][u(x,T;F_n) - u(x,T;F)]\,dx \\ &= J(F_n) - \int_\Omega [u(x,T;F_n) - u(x,T;F)]^2\,dx \\ &\quad - 2\int_\Omega [u(x,T;F) - u_T(x)]\Delta u_n(x,T;F)\,dx \end{aligned}$$

where $\Delta u_n(x,t;F) = u(x,T;F_n) - u(x,T;F)$ is the change of u corresponding to the change of the free term $\Delta F_n = F_n - F$. Thus, in view of (3.12) we have

$$J(F) \le J(F_n) - \iint\limits_{\Omega_T} \psi(x,t;F)\Delta F_n(x,t)\,dx\,dt.$$

Since $\psi \in L^2(\Omega_T)$, this implies the relation (3.17). To prove the convexity, we firstly note that

$$u\big(x,t;\gamma F_1 + (1-\gamma)F_2\big) = \gamma u(x,t;F_1) + (1-\gamma)u(x,t;F_2), \quad \text{for} \ \ \gamma \in [0,1].$$

Therefore, in view of the convexity of the quadratic function we obtain

$$\begin{aligned} J(\gamma F_1 + (1-\gamma)F_2) &= \int_0^T \Big[u\big(x,T,\gamma F_1 + (1-\gamma)F_2\big) - u_T(x)\Big]^2\,dx \\ &= \int_0^T \Big[\gamma\Big\{u(x,T;F_1) - u_T(x)\Big\} + (1-\gamma)\Big\{u(x,T;F_2) - u_T(x)\Big\}\Big]^2\,dx \\ &\le \gamma\int_0^T [u(x,T,F_1) - u_T(x)]^2\,dx + (1-\gamma)\int_0^T [u(x,T,F_2) - u_T(x)]^2\,dx \\ &= \gamma J(F_1) + (1-\gamma)J(F_2) \quad \text{for } \gamma \in [0,1]. \end{aligned}$$

This shows the convexity of J. Theorem is proved. □

Remark 1. In order to prove the existence in an unbounded set $\mathcal{F}$ incl. $\widehat{\mathcal{F}}$, it is sufficient to have the weak coercivity of $J(F)$. This is a difficult problem, because monotonicity methods in general fail for problems in integro-differential PDE. However, the boundedness assumption of $\mathcal{F}$ seems not very restrictive, because in practice some bound for F may be available.

4 Regularized Problem

In [5] we proved that in a particular case the solution of the inverse problem under consideration continuously depends on certain derivatives of the data.

This shows the ill-posedness of the problem in case the data have noise in L^2 space. We can easily incorporate Tikhonov regularization in quasi-solution. In this case we minimize the stabilized cost functional: find

$$F^* = \arg\min_{F\in\mathcal{F}} J_\alpha(F), \quad J_\alpha(F) = \alpha\|F\|^2_{L^2(\Omega_T)} + \|u(\cdot,T;F) - u_T\|^2_{L^2(\Omega)}.$$

Here $\alpha > 0$ is the regularization parameter that depends on the noise level of the data u_T. If we set here $\alpha = 0$, we get the original problem (3.1).

Theorem 3. *Let $\alpha > 0$ and $\mathcal{F}$ be a closed and convex subset of $\widehat{\mathcal{F}}$ (may be also $\mathcal{F} = \widehat{\mathcal{F}}$). Then the problem* (4.1) *has a unique solution in $\mathcal{F}$.*

Proof. Obviously the additional term $I(F) = \alpha\|F\|_{L^2(\Omega_T)}$ is strictly convex:

$$I(\gamma F_1 + (1-\gamma)F_2) < \gamma I(F_1) + (1-\gamma)I(F_2) \quad \forall\gamma\in(0,1),\ F_1,F_2\in\mathcal{F}$$

and weakly coercive, i.e., $I(F)\to\infty$ as $\|F\|_{L^2(\Omega_T)}\to\infty$. This makes the whole functional J_α strictly convex and weakly coercive. Moreover, it is easy to check that $I(F)$ is weakly sequentially lower semi-continuous. Since $J(F) = \|u(\cdot,T;F) - u_T\|^2_{L^2(\Omega)}$ is also weakly lower semi-continuous (this was shown in the proof of Theorem 2), the whole functional J_α is weakly lower semi-continuous. Now the assertion of the theorem follows from Weierstrass existence theorem [16, Section 2.5]. □

5 Auxiliary Estimates

Lemma 2. *The following estimate is valid with a constant C_0:*

$$\|\Delta u(\cdot,T;F)\|_{L^2(\Omega)} \le C_0\|\Delta F\|_{L^2(\Omega_T)}. \tag{5.1}$$

Proof. For the sake of shortness, we omit F in the list of arguments of Δu. Firstly, we prove this assertion in case $\|m\|_{L^1(0,T)}$ is small enough and the equation for Δu (3.3) contains an additional term, namely it has the form

$$\Delta u_t = A\Delta u - \sigma\Delta u - m * A\Delta u + \Delta F \quad \text{in } \Omega_T, \tag{5.2}$$

where σ is a sufficiently large number such that $\sigma - a(x) \ge \epsilon$ for any $x\in\Omega$. By Theorem 1, Δu belongs to $W_2^{2,1}(\Omega_T)$ and solves the problem (5.2), (3.4)–(3.6) in the classical sense. Let us multiply the equation (5.2) by Δu and integrate by parts taking into account the definition of A and the homogeneous boundary conditions (3.5), (3.6):

$$\begin{aligned}0 &= \iint\limits_{\Omega_T}\Big[\Delta u_t - (A-\sigma)\Delta u + m * A\Delta u - \Delta F\Big]\Delta u\,dx\,dt\\ &= \frac12\iint\limits_{\Omega_T}[\Delta u^2]_t\,dx\,dt + \iint\limits_{\Omega_T}\Big[\sum_{i,j=1}^n a_{ij}\Delta u_{x_j}\Delta u_{x_i} + (\sigma - a)\Delta u^2\Big]dx\,dt\end{aligned}$$

$$- \iint\limits_{\Omega_T} \Big[\sum_{i,j=1}^{n} a_{ij}(m * \Delta u_{x_j}) \Delta u_{x_i} - a(m * \Delta u)\Delta u \Big] dx\,dt$$
$$+ \iint\limits_{\Gamma_{2,T}} \vartheta \Delta u^2 \, d\Gamma \, dt - \iint\limits_{\Omega_T} \Delta F \Delta u \, dx\, dt.$$

In view of the homogeneous initial condition (3.4), this relation can be transformed to the form

$$\frac{1}{2}\int\limits_{\Omega} [\Delta u(x,T)]^2\, dx + \iint\limits_{\Gamma_{2,T}} \vartheta \Delta u^2 \, d\Gamma\, dt \tag{5.3}$$
$$+ \iint\limits_{\Omega_T} \Big[\sum_{i,j=1}^{n} a_{ij} \Delta u_{x_j} \Delta u_{x_i} + (\sigma - a)\Delta u^2 \Big] dx\, dt$$
$$= \iint\limits_{\Omega_T} \Big[\sum_{i,j=1}^{n} a_{ij}(m * \Delta u_{x_j}) \Delta u_{x_i} - a(m * \Delta u)\Delta u \Big] dx\,dt + \iint\limits_{\Omega_T} \Delta F \Delta u \, dx\, dt.$$

Due to the assumptions $\vartheta \geq 0$, (2.8) and $\sigma - a \geq \epsilon$, the left hand side of (5.3) can be estimated from below:

$$\frac{1}{2}\int\limits_{\Omega} [\Delta u(x,T)]^2\, dx + \iint\limits_{\Gamma_{2,T}} \vartheta \Delta u^2 \, dx\, dt$$
$$+ \iint\limits_{\Omega_T} \Big[\sum_{i,j=1}^{n} a_{ij} \Delta u_{x_j} \Delta u_{x_i} + (\sigma - a)\Delta u^2 \Big] dx\, dt$$
$$\geq \frac{1}{2}\int\limits_{\Omega} [\Delta u(x,T)]^2\, dx + \epsilon \iint\limits_{\Omega_T} \big[|\nabla \Delta u|^2 + \Delta u^2 \big]\, dx\, dt =: I^2. \tag{5.4}$$

The right-hand side of (5.3) is estimated from above by means of the Cauchy–Schwarz inequality:

$$\iint\limits_{\Omega_T} \Big[\sum_{i,j=1}^{n} a_{ij}(m * \Delta u_{x_j}) \Delta u_{x_i} - a(m * \Delta u)\Delta u \Big] dx\,dt + \iint\limits_{\Omega_T} \Delta F \Delta u \, dx\, dt$$
$$\leq \bar{C}_1 \Big[\sum_{i,j=1}^{n} \big\| m * \Delta u_{x_j} \big\|_{L^2(\Omega_T)} \|\Delta u_{x_i}\|_{L^2(\Omega_T)}$$
$$+ \|m * \Delta u\|_{L^2(\Omega_T)} \|\Delta u\|_{L^2(\Omega_T)} \Big] + \|\Delta F\|_{L^2(\Omega_T)} \|\Delta u\|_{L^2(\Omega_T)} \tag{5.5}$$

where $\bar{C}_1$ is a constant depending on the coefficients a_{ij} and a. For the convolution terms we apply the Young's inequality in the space $L^2(\Omega_T) = L^2((0,T); L^2(\Omega))$. This yields

$$\|m * \Delta u_{x_j}\|_{L^2(\Omega_T)} \leq \|m\|_{L^1(0,T)} \|\Delta u_{x_j}\|_{L^2(\Omega_T)}, \quad j = 1, \ldots, n,$$
$$\|m * \Delta u\|_{L^2(\Omega_T)} \leq \|m\|_{L^1(0,T)} \|\Delta u\|_{L^2(\Omega_T)}. \tag{5.6}$$

Using (5.4)–(5.6) in (5.3) we obtain

$$I^2 \le \bar{C}_1 \|m\|_{L^1(0,T)} \Big[\sum_{i,j=1}^{n} \|\Delta u_{x_j}\|_{L^2(\Omega_T)} \|\Delta u_{x_i}\|_{L^2(\Omega_T)} + \|\Delta u\|^2_{L^2(\Omega_T)} \Big] + \|\Delta F\|_{L^2(\Omega_T)} \|\Delta u\|_{L^2(\Omega_T)}.$$

Further, we use the inequalities

$$\|\Delta u_{x_i}\|_{L^2(\Omega_T)} \le \||\nabla \Delta u|\|_{L^2(\Omega_T)}, \quad i = 1, \dots, n,$$

and definition of I (see (5.4)). We have

$$I^2 \le \bar{C}_1 \|m\|_{L^1(0,T)} \Big[n^2 \||\nabla \Delta u|\|^2_{L^2(\Omega_T)} + \|\Delta u\|^2_{L^2(\Omega_T)} \Big] + \|\Delta F\|_{L^2(\Omega_T)} \|\Delta u\|_{L^2(\Omega_T)} \le \frac{\bar{C}_1 n^2 \|m\|_{L^1(0,T)}}{\epsilon} I^2 + \frac{1}{\sqrt{\epsilon}} \|\Delta F\|_{L^2(\Omega_T)} I.$$

Therefore, in case m satisfies the smallness condition

$$\|m\|_{L^1(0,T)} \le \frac{\epsilon}{2\bar{C}_1 n^2}, \tag{5.7}$$

we obtain $I^2 \le \frac{2}{\sqrt{\epsilon}} \|\Delta F\|_{L^2(\Omega_T)} I$ that yields $I \le \frac{2}{\sqrt{\epsilon}} \|\Delta F\|_{L^2(\Omega_T)}$. Observing that $\|\Delta u(\cdot, T)\|_{L^2(\Omega)} \le \sqrt{2} I$, from the latter inequality we deduce the estimate (5.1) with the constant $C_0 = 2\sqrt{2}/\epsilon$.

Now let us return to the original problem (3.3)–(3.6) without the additional σ-term and arbitrarily large m. Define the following function: $\Delta u_\sigma(x,t) = e^{-\sigma t} \Delta u(x,t)$ where $\sigma \in \mathbb{R}$. It is easy to check that Δu_σ solves the following problem:

$$\begin{aligned}
&\Delta u_{\sigma,t} = A \Delta u_\sigma - \sigma \Delta u_\sigma - m_\sigma * A \Delta u_\sigma + \Delta F_\sigma \quad \text{in } \Omega_T, \\
&\Delta u_\sigma = 0 \quad \text{in } \Omega \times \{0\}, \\
&\Delta u_\sigma = 0 \quad \text{in } \Gamma_{1,T}, \\
&- \nu_A \cdot \nabla \Delta u_\sigma + m_\sigma * \nu_A \cdot \nabla \Delta u_\sigma = \vartheta \Delta u_\sigma \quad \text{in } \Gamma_{2,T}
\end{aligned}$$

where $m_\sigma(t) = e^{-\sigma t} m(t)$ and $\Delta F_\sigma(x,t) = e^{-\sigma t} \Delta F(x,t)$. Clearly, there exists a sufficiently large σ such that m_σ satisfies the condition (5.7) and the inequality $\sigma - a(x) \ge \epsilon$ is valid for $x \in \Omega$. Therefore, the first part of the proof applies to the function Δu_σ. This means that the estimate

$$\|\Delta u_\sigma(\cdot, T)\|_{L^2(\Omega)} \le \frac{2\sqrt{2}}{\epsilon} \|\Delta F_\sigma\|_{L^2(\Omega_T)} \tag{5.8}$$

is valid. Finally, in view of $\Delta u_\sigma(x,T) = e^{-\sigma T} \Delta u(x,T)$ and $|\Delta F_\sigma(x,t)| \le |\Delta F(x,t)|$, from (5.8) we obtain the desired estimate (5.1) with the constant $C_0 = 2\sqrt{2} e^{\sigma T}/\epsilon$. Lemma 2 is proved. □

Further, let us estimate the difference of solutions of the adjoint problems

$$\Delta \psi(x,t;F) = \psi(x,t;F + \Delta F) - \psi(x,t;F).$$

Lemma 3. *The following estimate is valid with a constant* C_1:

$$\|\Delta\psi(\cdot,\cdot;F)\|_{L^2(\Omega_T)} \le C_1\|\Delta F\|_{L^2(\Omega_T)}. \tag{5.9}$$

Proof. Proof is similar to the proof of the previous lemma. Observing (3.7)–(3.10) we see that the problem for $\Delta\psi(x,t;F)$ has the following form:

$$\Delta\psi_t(x,t;F) = -A\Delta\psi(x,t;F) + \int_t^T m(\tau-t)A\Delta\psi(x,\tau;F)\,d\tau \quad \text{in } \Omega_T, \tag{5.10}$$

$$\Delta\psi(x,T;F) = 2\Delta u(x,T;F) \quad \text{in } \Omega, \tag{5.11}$$

$$\Delta\psi(x,t;F) = 0 \quad \text{in } \Gamma_{1,T}, \tag{5.12}$$

$$\begin{aligned} &-\nu_A\cdot\nabla\Delta\psi(x,t;F) + \int_t^T m(\tau-t)\nu_A\cdot\nabla\Delta\psi(x,\tau;F)\,d\tau \\ &\quad = \vartheta\Delta\psi(x,t;F) \quad \text{in } \Gamma_{2,T}. \end{aligned} \tag{5.13}$$

We start by proving the assertion in case $\|m\|_{L^1(0,T)}$ is small enough and the equation (3.3) contains an additional term, namely it has the form

$$\begin{aligned} \Delta\psi_t(x,t;F) &= -A\Delta\psi(x,t;F) + \sigma\Delta\psi(x,t;F) \\ &\quad + \int_t^T m(\tau-t)A\Delta\psi(x,\tau;F)\,d\tau \quad \text{in } \Omega_T, \end{aligned} \tag{5.14}$$

where σ is again sufficiently large, i.e. $\sigma - a(x) \ge \epsilon$ for any $x\in\Omega$. Since $\Delta u \in W_2^{2,1}(\Omega_T)$, by the trace theorem it holds $\Delta u|_{t=T} \in H^1(\Omega)$. Moreover, one can immediately check that the time-inverted function $\Delta\psi(x,T-t;F)$ satisfies a problem of the form (2.1)–(2.4) with an homogeneous equation, homogeneous boundary conditions and the initial condition $2\Delta u(x,T;F)$. Therefore, applying Theorem 1 we see that the function $\Delta\psi(x,t;F)$ belongs to $W_2^{2,1}(\Omega_T)$ and satisfies the problem (5.14), (5.11), (5.12), (5.13) in the classical sense. For the sake of shortness we omit the argument F of $\Delta\psi$ and Δu in forthcoming computations. Multiplying (5.14) by $\Delta\psi$ and integrating by parts we obtain

$$\begin{aligned} 0 &= \iint_{\Omega_T}\Big[\Delta\psi_t + (A-\sigma)\Delta\psi - \int_t^T m(\tau-t)A\Delta\psi(x,\tau)\,d\tau\Big]\Delta\psi\,dx\,dt \\ &= \frac{1}{2}\iint_{\Omega_T}[\Delta\psi^2]_t\,dx\,dt - \iint_{\Omega_T}\Big[\sum_{i,j=1}^n a_{ij}\Delta\psi_{x_j}\Delta\psi_{x_i} + (\sigma-a)\Delta\psi^2\Big]dx\,dt \\ &\quad + \iint_{\Omega_T}\Big[\sum_{i,j=1}^n a_{ij}(x)\int_t^T m(\tau-t)\Delta\psi_{x_j}(x,\tau)\,d\tau\Delta\psi_{x_i}(x,t) \\ &\quad - a(x)\int_t^T m(\tau-t)\Delta\psi(x,\tau)\,d\tau\Delta\psi(x,t)\Big]dx\,dt - \iint_{\Gamma_{2,T}}\vartheta\Delta\psi^2\,d\Gamma\,dt. \end{aligned}$$

Observing the final condition (5.11) and rearranging the terms we get

$$\frac{1}{2}\int_\Omega[\Delta\psi(x,0)]^2\,dx + \iint_{\Gamma_{2,T}}\vartheta\Delta\psi^2\,d\Gamma\,dt \tag{5.15}$$

$$+\iint\limits_{\Omega_T}\Big[\sum_{i,j=1}^{n}a_{ij}\Delta\psi_{x_j}\Delta\psi_{x_i}+(\sigma-a)\Delta\psi^2\Big]dx\,dt$$
$$=\iint\limits_{\Omega_T}\Big[\sum_{i,j=1}^{n}a_{ij}(x)\int_t^T m(\tau-t)\Delta\psi_{x_j}(x,\tau)\,d\tau\Delta\psi_{x_i}(x,t)$$
$$-a(x)\int_t^T m(\tau-t)\Delta\psi(x,\tau)\,d\tau\Delta\psi(x,t)\Big]dx\,dt+\frac{1}{2}\int_\Omega[\Delta u(x,T)]^2\,dx. \quad (5.16)$$

The left-hand side of (5.15) is estimated from below:

$$\frac{1}{2}\int\limits_\Omega[\Delta\psi(x,0)]^2\,dx+\iint\limits_{\Gamma_{2,T}}\vartheta\Delta\psi^2\,d\Gamma\,dt+\iint\limits_{\Omega_T}\Big[\sum_{i,j=1}^{n}a_{ij}\Delta\psi_{x_j}\Delta\psi_{x_i}$$
$$+(\sigma-a)\Delta\psi^2\Big]dx\,dt\ge\epsilon\big[\||\nabla\Delta\psi|\|^2_{L^2(\Omega_T)}+\|\Delta\psi\|^2_{L^2(\Omega_T)}\big]=:S^2. \quad (5.17)$$

For the right-hand side of (5.15) we use the Cauchy–Schwarz inequality:

$$\iint\limits_{\Omega_T}\Big[\sum_{i,j=1}^{n}a_{ij}(x)\int_t^T m(\tau-t)\Delta\psi_{x_j}(x,\tau)\,d\tau\Delta\psi_{x_i}(x,t)$$
$$-a(x)\int_t^T m(\tau-t)\Delta\psi(x,\tau)\,d\tau\Delta\psi(x,t)\Big]dx\,dt+\frac{1}{2}\int_\Omega[\Delta u(x,T)]^2\,dx$$
$$\le\hat{C}_1\Big[\sum_{i,j=1}^{n}\Big\|\int_t^T m(\tau-t)\Delta\psi_{x_j}(x,\tau)\,d\tau\Big\|_{L^2(\Omega_T)}\|\Delta\psi_{x_i}\|_{L^2(\Omega_T)}$$
$$+\Big\|\int_t^T m(\tau-t)\Delta\psi(x,\tau)\,d\tau\Big\|_{L^2(\Omega_T)}\|\Delta\psi\|_{L^2(\Omega_T)}\Big]+\frac{1}{2}\|\Delta u(\cdot,T)\|^2_{L^2(\Omega)} \quad (5.18)$$

with some constant $\hat{C}_1$. It is easy to check by means of the change of variables of integration that

$$\Big\|\int_t^T m(\tau-t)v(x,\tau)\,d\tau\Big\|_{L^2(\Omega_T)}=\|m*v\|_{L^2(\Omega_T)}\ \text{ for any }\ v.$$

Therefore, using the Young's inequality we get

$$\Big\|\int_t^T m(\tau-t)\Delta\psi_{x_j}(x,\tau)\,d\tau\Big\|_{L^2(\Omega_T)}\le\|m\|_{L^1(0,T)}\|\Delta\psi_{x_j}\|_{L^2(\Omega_T)},$$
$$\Big\|\int_t^T m(\tau-t)\Delta\psi(x,\tau)\,d\tau\Big\|_{L^2(\Omega_T)}\le\|m\|_{L^1(0,T)}\|\Delta\psi\|_{L^2(\Omega_T)}. \quad (5.19)$$

By means of (5.17)–(5.19) from (5.17) we obtain the relation

$$S^2\le\hat{C}_1\|m\|_{L^1(0,T)}\Big[\sum_{i,j=1}^{n}\|\Delta\psi_{x_j}\|_{L^2(\Omega_T)}\|\Delta\psi_{x_i}\|_{L^2(\Omega_T)}+\|\Delta\psi\|^2_{L^2(\Omega_T)}\Big]$$
$$+\frac{1}{2}\|\Delta u(\cdot,T)\|^2_{L^2(\Omega)}.$$

Like in the proof of Lemma 3 from this relation and the definition of S we deduce the estimate $\|\Delta\psi\|_{L^2(\Omega_T)} \leq \frac{1}{\sqrt{\epsilon}}\|\Delta u(\cdot,T)\|_{L^2(\Omega)}$ provided m satisfies the inequality

$$\|m\|_{L^1(0,T)} \leq \frac{\epsilon}{2\hat{C}_1 n^2}. \tag{5.20}$$

Further, applying Lemma 2 to the obtained estimate we get (5.9) with the constant $C_1 = C_0/\sqrt{\epsilon}$.

Finally, let us consider the original problem for $\Delta\psi$ without the additional σ-term and arbitrarily large m. Define $\Delta\psi_\sigma(x,t) = e^{-\sigma(T-t)}\Delta u(x,t)$ with $\sigma \in \mathbb{R}$. Then $\Delta\psi_\sigma$ solves the following problem:

$$\begin{aligned}
&\Delta\psi_{\sigma,t}(x,t) = -A\Delta\psi_\sigma(x,t) + \int_t^T m_\sigma(\tau-t)A\Delta\psi_\sigma(x,\tau)\,d\tau \quad \text{in } \Omega_T,\\
&\Delta\psi_\sigma(x,T) = 2\Delta u(x,T) \quad \text{in } \Omega, \quad \Delta\psi_\sigma(x,t) = 0 \quad \text{in } \Gamma_{1,T},\\
&-\nu_A\cdot\nabla\Delta\psi_\sigma(x,t) + \int_t^T m_\sigma(\tau-t)\nu_A\cdot\nabla\Delta\psi_\sigma(x,\tau;)\,d\tau = \vartheta\Delta\psi_\sigma(x,t) \quad \text{in } \Gamma_{2,T},
\end{aligned}$$

where $m_\sigma(t) = e^{-\sigma t}m(t)$ again. There exists a sufficiently large σ such that m_σ satisfies the condition (5.20) and the inequality $\sigma - a(x) \geq \epsilon$ is valid for $x \in \Omega$. Thus, applying the first part of the proof to $\Delta\psi_\sigma$ we have

$$\|\Delta\psi_\sigma\|_{L^2(\Omega_T)} \leq \frac{C_0}{\sqrt{\epsilon}}\|\Delta F\|_{L^2(\Omega_T)}.$$

Since $\|\Delta\psi_\sigma\|_{L^2(\Omega_T)} \geq e^{-\sigma T}\|\Delta\psi\|_{L^2(\Omega_T)}$ we reach the estimate (5.9) with the constant $C_1 = C_0 e^{\sigma T}/\sqrt{\epsilon}$. Lemma 3 is proved. □

6 Frechet Derivative and Gradient Method

It follows from Lemma 2 with (3.2) that the functional J is Frechet differentiable in $L^2(\Omega_T)$. Moreover, according to Lemma 1, $J'(F)$ is identical to the element $\psi(F) = \psi(x,t;F)$ in $L^2(\Omega_T)$, i.e. it holds

$$J'(F)\tilde{F} = \big(\psi(F),\tilde{F}\big)_{L^2(\Omega_T)} = \iint_{\Omega_T} \psi(x,t;F)\tilde{F}(x,t)\,dx\,dt \quad \forall \tilde{F} \in L^2(\Omega_T).$$

Similarly, J_α is Frechet differentiable in $L^2(\Omega_T)$ and

$$\begin{aligned}
J'_\alpha(F)\tilde{F} &= \big(2\alpha F + \psi(F),\tilde{F}\big)_{L^2(\Omega_T)}\\
&= \iint_{\Omega_T} (2\alpha F(x,t) + \psi(x,t;F))\tilde{F}(x,t)\,dx\,dt \quad \forall \tilde{F} \in L^2(\Omega_T).
\end{aligned} \tag{6.1}$$

Therefore, gradient-type methods can be used to solve the minimization problems (3.1) and (4.1). These methods must be combined by proper projection techniques to get minimum in the subset $\mathcal{F}$. However, it is possible to simplify the minimization procedure in case the structure of the subspace $\widehat{\mathcal{F}}$ is simple. In particular, global optimization can be used if $\mathcal{F} = \widehat{\mathcal{F}}$. To this end, let us consider the cases 1–3 introduced in Section 3.

Case 1. We introduce the functional $\Phi_{1,\alpha}(w) = J_\alpha(\varkappa w)$ with $\alpha \geq 0$ and the set $\mathcal{W}_1 = \{w \in L^2(\Omega)\colon \varkappa w \in \mathcal{F}\}$. Then the problem (4.1) (in case $\alpha = 0$ the problem (3.1)) can be rewritten as follows:

$$\text{find } w^* = \arg \min_{w \in \mathcal{W}_1} \Phi_{1,\alpha}(w). \tag{6.2}$$

In particular, when $\mathcal{F} = \widehat{\mathcal{F}}$, it holds $\mathcal{W}_1 = L^2(\Omega)$ and we have a global minimization problem. Since J_α is Frechet differentiable, $\Phi_{1,\alpha}$ is also Frechet differentiable. Moreover, from (6.1) we deduce

$$J'_\alpha(\varkappa w)\varkappa\tilde{w} = \int_\Omega \Big[\int_0^T [2\alpha w(x)\varkappa(t) + \psi(x,t,\varkappa w)]\varkappa(t)\, dt \Big] \tilde{w}(x)\, dx.$$

This shows that $\Phi'_{1,\alpha}(w)$ is identical to the element $\int_0^T [2\alpha w(x)\varkappa(t) + \psi(x,t,\varkappa w)]\varkappa(t)dt$ of $L^2(\Omega)$, that is

$$\Phi'_{1,\alpha}(w)\tilde{w} = \Big(\int_0^T [2\alpha w\varkappa(t) + \psi(\cdot,t,\varkappa w)]\varkappa(t)\, dt, \tilde{w} \Big)_{L^2(\Omega)} \quad \forall \tilde{w} \in L^2(\Omega).$$

Using Cauchy–Schwarz inequality and Lemma 3 we estimate

$$\begin{aligned}
&\|\Phi'_{1,\alpha}(w + \Delta w) - \Phi'_{1,\alpha}(w)\|_{L^2(\Omega)} \\
&= \Big[\int_\Omega \Big\{ \int_0^T [2\alpha\Delta w(x)\varkappa(t) + \psi(x,t,\varkappa(w+\Delta w)) - \psi(x,t,\varkappa w)]\, \varkappa(t)dt \Big\}^2 dx \Big]^{1/2} \\
&\leq \|2\alpha\Delta w(x)\varkappa(t) + \psi(x,t,\varkappa(w+\Delta w)) - \psi(x,t,\varkappa w)\|_{L^2(\Omega_T)} \|\varkappa\|_{L^2(0,T)} \\
&\leq (2\alpha + C_1)\|\varkappa\Delta w\|_{L^2(\Omega_T)}\|\varkappa\|_{L^2(0,T)} = (2\alpha + C_1)\|\varkappa\|^2_{L^2(0,T)}\|\Delta w\|_{L^2(\Omega)}.
\end{aligned}$$

This implies that $\Phi'_{1,\alpha}$ is uniformly Lipschitz-continuous, i.e.

$$\|\Phi'_{1,\alpha}(w + \Delta w) - \Phi'_{1,\alpha}(w)\|_{L^2(\Omega)} \leq L_\alpha \|\Delta w\|_{L^2(\Omega)} \tag{6.3}$$

where $L_\alpha = (2\alpha + C_1)\|\varkappa\|^2_{L^2(0,T)}$.

The cases 2 and 3 can be treated in a similar manner. Let us summarize the results in these cases.

Case 2. Define $\Phi_{2,\alpha}(w) = J_\alpha(\varkappa w)$ with $\alpha \geq 0$ and the set $\mathcal{W}_2 = \{w \in L^2(S_T)\colon \varkappa w \in \mathcal{F}\}$. If $\mathcal{F} = \widehat{\mathcal{F}}$ then $\mathcal{W}_2 = L^2(S_T)$. The problem (4.1) can be rewritten in the form: find $w^* = \arg \min_{w \in \mathcal{W}_2} \Phi_{2,\alpha}(w)$. The functional $\Phi_{2,\alpha}$ is Frechet differentiable, $\Phi'_{2,\alpha}(w)$ is identical to the element $\int_0^l [2\alpha w(\overline{x},t)\varkappa(x_n) + \psi(x,t,\varkappa w)]\varkappa(x_n)\, dx_n$ of $L^2(S_T)$ and the uniform Lipschitz-estimate

$$\|\Phi'_{2,\alpha}(w + \Delta w) - \Phi'_{2,\alpha}(w)\|_{L^2(S_T)} \leq L_\alpha \|\Delta w\|_{L^2(S_T)} \tag{6.4}$$

is valid with $L_\alpha = (2\alpha + C_1)\|\varkappa\|^2_{L^2(0,l)}$.

Case 3. Let $\Phi_{3,\alpha}(w) = J_\alpha(\sum_{j=1}^N w_j \varkappa_j)$ with $\alpha \geq 0$ and $\mathcal{W}_3 = \{w \in \mathbb{R}^N\colon \sum_{j=1}^N w_j \varkappa_j \in \mathcal{F}\}$. If $\mathcal{F} = \widehat{\mathcal{F}}$ then $\mathcal{W}_2 = \mathbb{R}^N$. The problem (4.1) admits the following form: find $w^* = \arg\min_{w \in \mathcal{W}_3} \Phi_{3,\alpha}(w)$. The functional $\Phi_{3,\alpha}$ is Frechet differentiable, $\Phi'_{3,\alpha}(w)$ is identical to the element $(\iint_{\Omega_T}[2\alpha \sum_{l=1}^N w_l \varkappa_l(x,t) + \psi(x,t,\sum_{l=1}^N w_l \varkappa_l)]\varkappa_j(x,t)\,dx\,dt)_{j=1,\dots,N}$ of $\mathbb{R}^N$ and the estimate

$$\|\Phi'_{3,\alpha}(w + \Delta w) - \Phi'_{3,\alpha}(w)\|_{\mathbb{R}^N} \leq L_\alpha \|\Delta w\|_{\mathbb{R}^N} \tag{6.5}$$

with $L_\alpha = (2\alpha + C_1)\sum_{j=1}^N \|\varkappa_j\|^2_{L^2(\Omega_T)}$ is valid.

In the following, let Φ_α be one of the functionals $\Phi_{j,\alpha}$, $j = 1, 2, 3$, defined above and $\mathcal{W}$ be the corresponding set of admissible solutions $\mathcal{W}_j$. Then we consider the problem

$$\text{find} \;\; w^* = \arg\min_{w \in \mathcal{W}} \Phi_\alpha(w). \tag{6.6}$$

For the sake of simplicity, we assume that $\mathcal{F} = \widehat{\mathcal{F}}$. This means that we consider the unconstrained minimization and $\mathcal{W}$ is $L^2(\Omega)$, $L^2(S_T)$ and $\mathbb{R}^N$ in the cases 1, 2 and 3, respectively. Let $w_0 \in \mathcal{W}$ be an initial guess and compute the successive approximations by means of the gradient method

$$w_{k+1} = w_k - c_k \Phi'_\alpha(w_k), \quad k = 0, 1, 2, \dots \tag{6.7}$$

with steps $c_k > 0$. Let us perform a little analysis for this iteration process following partially the example of [2].

Lemma 4. *For any $\alpha \geq 0$ it holds*

$$|\Phi_\alpha(w_{k+1}) - \Phi_\alpha(w_k) - \Phi'_\alpha(w_k)(w_{k+1} - w_k)| \leq \frac{L_\alpha}{2}\|w_{k+1} - w_k\|^2. \tag{6.8}$$

Proof. Using the relation

$$\Phi_\alpha(w_{k+1}) - \Phi_\alpha(w_k) = \int_0^1 \Phi'_\alpha(w_k + \tau(w_{k+1} - w_k))(w_{k+1} - w_k)\,d\tau$$

and the estimates (6.3)–(6.5) we deduce

$$\begin{aligned}
&|\Phi_\alpha(w_{k+1}) - \Phi_\alpha(w_k) - \Phi'_\alpha(w_k)(w_{k+1} - w_k)| \\
&\quad = \left| \int_0^1 [\Phi'_\alpha(w_k + \tau(w_{k+1} - w_k)) - \Phi'_\alpha(w_k)](w_{k+1} - w_k)\,d\tau \right| \\
&\quad \leq L_\alpha \|w_{k+1} - w_k\|^2 \int_0^1 \tau\,d\tau = \frac{L_\alpha}{2}\|w_{k+1} - w_k\|^2.
\end{aligned}$$

This proves (6.8). □

Theorem 4. *Let $\alpha \geq 0$ and $\delta \leq c_k \leq 2/L_\alpha - \delta$ for any $k = 0, 1, 2, \dots$ where δ is some number in the half-interval $(0, 1/L_\alpha]$. Then the sequence $\Phi_\alpha(w_k)$ is* *monotonically decreasing, has a limit and the following relations are valid with* $q_k = c_k - L_\alpha c_k^2/2 \geq \delta - L_\alpha \delta^2/2 > 0$:

$$\Phi_\alpha(w_k) - \Phi_\alpha(w_{k+1}) \geq q_k \|\Phi'_\alpha(w_k)\|^2, \quad k = 0, 1, 2, \ldots, \tag{6.9}$$

$$\Phi'_\alpha(w_k) \to 0 \quad \text{as } k \to \infty, \tag{6.10}$$

$$\|w_{k+1} - w_k\|^2 \leq \frac{c_k^2}{q_k} \big[\Phi_\alpha(w_k) - \Phi_\alpha(w_{k+1})\big], \quad k = 0, 1, 2, \ldots. \tag{6.11}$$

Proof. Due to (6.7) it hold $\|w_{k+1} - w_k\|^2 \leq c_k^2 \|\Phi'_\alpha(w_k)\|^2$ and

$$\Phi'_\alpha(w_k)(w_{k+1} - w_k) = \big(\Phi'_\alpha(w_k), -c_k \Phi'_\alpha(w_k)\big)_{\mathcal{W}} = -c_k \|\Phi'_\alpha(w_k)\|^2.$$

Thus, by means of (6.8) we get

$$\begin{aligned}
&\Phi_\alpha(w_{k+1}) - \Phi_\alpha(w_k) + c_k \|\Phi'_\alpha(w_k)\|^2 \\
&\quad \leq \big|\Phi_\alpha(w_{k+1}) - \Phi_\alpha(w_k) + c_k \|\Phi'_\alpha(w_k)\|^2\big| \leq \frac{L_\alpha c_k^2}{2} \|\Phi'_\alpha(w_k)\|^2.
\end{aligned}$$

This yields $\Phi_\alpha(w_k) - \Phi_\alpha(w_{k+1}) \geq \big(c_k - \frac{L_\alpha c_k^2}{2}\big) \|\Phi'_\alpha(w_k)\|^2$, i.e. (6.9). Due to $q_k > 0$, the relation (6.9) implies that $\Phi_\alpha(w_k)$ is monotonically decreasing and since $\Phi_\alpha(w)$ has the lower bound 0, the sequence $\Phi_\alpha(w_k)$ converges. Further, since the sequence q_k has the positive lower bound $\delta - \frac{L_\alpha \delta^2}{2}$ and the left hand side of (6.9) converges to zero, we obtain (6.10). Finally, estimating (6.7) we have $\|w_{k+1} - w_k\|^2 = c_k^2 \|\Phi'_\alpha(w_k)\|^2$. Using here (6.9) we obtain (6.11). Theorem is proved. □

Clearly, the highest decrease rate of $\Phi_\alpha(w_k)$ is achieved in case $c_k = 1/L_\alpha$ when q_k has the biggest value $q_k = 1/2L_\alpha$.

Theorem 5. *Let* $\alpha > 0$ *and* c_k *be chosen as in Theorem* 4*. Then the sequence* w_k *strongly converges to the unique solution of the minimization problem* (6.6)*.*

Proof. The existence of the unique solution for the minimization problem immediately follows from Theorem 3 and the definitions of Φ_α. Moreover, since J_α is weakly sequentially lower semi-continuous, strictly convex and weakly coercive (see the proof of Theorem 3), the same properties are valid also for Φ_α. It is well-known that under such properties every minimizing sequence of Φ_α weakly converges to the minimum point w^*. Thus, firstly, let us show that w_k is a minimizing sequence, i.e. $\Phi_\alpha(w_k) \to \Phi_\alpha(w^*)$.

Note that the sequence w_k is bounded. Indeed, otherwise there exists a subsequence w_{k_i} such that $\|w_{k_i}\| \to \infty$ and by the weak coercitivity it holds $\Phi_\alpha(w_{k_i}) \to \infty$ which contradicts to the statement of Theorem 4 that $\Phi_\alpha(w_k)$ is monotonically decreasing.

Since Φ_α is convex, its Frechet derivative is monotone, i.e.

$$\big[\Phi'_\alpha(\widetilde{w}) - \Phi'_\alpha(w)\big](\widetilde{w} - w) \geq 0 \quad \forall w, \widetilde{w} \in \mathcal{W}. \tag{6.12}$$

Let us choose some $\tau \in (0,1)$. Observing that it holds $\Phi'_\alpha(w^*) = 0$ in the global minimum point w^* and applying (6.12) with $w = w^*$ and $\widetilde{w} = w^* + \tau(w_k - w^*)$ we have

$$\liminf_{k\to\infty} \Phi'_\alpha(w^* + \tau(w_k - w^*))(w_k - w^*) = \frac{1}{\tau} \liminf_{k\to\infty} \big[\Phi'_\alpha(w^* + \tau(w_k - w^*)) - \Phi'_\alpha(w^*)\big](w^* + \tau(w_k - w^*) - w^*) \geq 0. \quad (6.13)$$

On the other hand, it holds $\lim_{k\to\infty} \Phi'_\alpha(w_k)(w_k - w^*) = 0$ because of the boundedness of w_k and the relation (6.10). Thus, using (6.12) with $w = w_k$ and $\widetilde{w} = w^* + \tau(w_k - w^*)$ we obtain

$$\limsup_{k\to\infty} \Phi'_\alpha(w^* + \tau(w_k - w^*))(w_k - w^*) = \frac{1}{1-\tau} \limsup_{k\to\infty} \big[\Phi'_\alpha(w^* + \tau(w_k - w^*)) - \Phi'_\alpha(w_k)\big](w_k - w^* - \tau(w_k - w^*)) \leq 0. \quad (6.14)$$

The estimates (6.13) and (6.14) imply $\limsup_{k\to\infty} v_k \leq 0 \leq \liminf_{k\to\infty} v_k$ for the sequence $v_k = \Phi'_\alpha(w^* + \tau(w_k - w^*))(w_k - w^*)$. Hence,

$$\lim_{k\to\infty} \Phi'_\alpha(w^* + \tau(w_k - w^*))(w_k - w^*) = 0. \quad (6.15)$$

Further, writing

$$\Phi_\alpha(w_k) - \Phi_\alpha(w^*) = \int_0^1 \Phi'_\alpha(w^* + \tau(w_k - w^*))(w_k - w^*)\, d\tau$$

and using (6.15) we obtain $\Phi_\alpha(w_k) - \Phi_\alpha(w^*) \to 0$. This shows that w_k is a minimizing sequence. Consequently, $w_k \rightharpoonup w^*$.

Now let us prove the assertion of the Theorem $w_k \to w^*$. In case 3 this is evident, because $\mathcal{W}$ is of finite dimension. Thus, let us study the cases 1 and 2. Then it holds $\Phi_\alpha(w) = \alpha\nu\|w\|^2 + \Phi_0(w)$ where ν is a positive constant ($\nu = \int_0^T \varkappa^2(t)\, dt$ in case 1 and $\nu = \int_0^l \varkappa^2(x_n)\, dx_n$ in case 2). Since the norm is weakly lower sequentially semicontinuous, the relation $w_k \rightharpoonup w^*$ implies

$$\|w^*\|^2 \leq \liminf_{k\to\infty} \|w_k\|^2. \quad (6.16)$$

On the other hand, since $\Phi_\alpha(w_k)$ converges to $\Phi_\alpha(w^*)$ and $\Phi_0(w)$ is weakly lower sequentially semicontinuous and we obtain

$$\begin{aligned}
\limsup_{k\to\infty} \|w_k\|^2 &= \frac{1}{\alpha\nu} \limsup_{k\to\infty} \big[\Phi_\alpha(w_k) - \Phi_0(w_k)\big] \\
&= \frac{1}{\alpha\nu} \Big\{ \lim_{k\to\infty} \Phi_\alpha(w_k) + \limsup_{k\to\infty} [-\Phi_0(w_k)] \Big\} \\
&= \frac{1}{\alpha\nu} \Big\{ \Phi_\alpha(w^*) - \liminf_{k\to\infty} \Phi_0(w_k) \Big\} \\
&\leq \frac{1}{\alpha\nu} \Big\{ \Phi_\alpha(w^*) - \Phi_0(w^*) \Big\} = \|w^*\|^2. \quad (6.17)
\end{aligned}$$

Putting together (6.16) and (6.17) we get $\limsup_{k\to\infty} \|w_k\|^2 \leq \|w^*\|^2 \leq \liminf_{k\to\infty} \|w_k\|^2$. This gives $\lim_{k\to\infty} \|w_k\|^2 = \|w^*\|^2$. Since in an Hilbert space the weak convergence and the convergence of norms implies the strong convergence, we prove $w_k \to w^*$. The proof is complete. □

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