Some Notes on Matrix Transforms of Summability Domains of Cesàro Matrices

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Abstract. In this paper sufficient conditions for a matrix \( M = (m_{nk}) \) (\( m_{nk} \) are Cesàro numbers \( A_{n-k}^s, s \in \mathbb{C} \) if \( k \leq n \) and \( m_{nk} = 0 \) if \( k > n \)) to be a transform from the summability domain of the Cesàro method \( C_\alpha \) into the summability domain of another Cesàro method \( C_\beta \), where \( \alpha, \beta \in \mathbb{C}\backslash\{-1, -2, \ldots\} \), are found.

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1 Introduction

Let \( C_\alpha = (c_{nk}), \alpha \in \mathbb{C}\backslash\{-1, -2, \ldots\} \), be a series-to-sequence Cesàro method, i.e. (see [4] or [5])

\[
c_{nk} = \begin{cases} \frac{A_{n-k}^\alpha}{A_n^\alpha} & (k \leq n), \\ 0 & (k > n), \end{cases}
\]

where \( A_n^\alpha = \binom{n+\alpha}{n} \) are Cesàro numbers. Throughout this paper we assume that summation indices run from 0 to \( \infty \) unless otherwise specified. A series \( x := \sum x_k \) is said to be \( C_\alpha \)-summable if the sequence \( C_\alpha x = (C_n^\alpha x) \) is convergent, where

\[
C_n^\alpha x = \sum_{k=0}^{n} c_{nk}x_k.
\]

We denote the domain of all \( C_\alpha \)-summable series by \( c_{C_\alpha} \), i.e.

\[
c_{C_\alpha} := \left\{ x = (x_n) \mid \lim_{n \to \infty} C_n^\alpha x \text{ exists} \right\}.
\]

In [1, 3, 10] necessary and sufficient conditions for a matrix \( M \) with real or complex entries to be a transform from \( c_{C_\alpha} \) into \( c_{C_\beta} \) for \( \alpha, \beta \in \mathbb{R} \) or \( \alpha, \beta \in \mathbb{C}\backslash\{-1, -2, \ldots\} \) are described. The summability domains (and the subsets of
summability domains) of different Cesàro methods are compared in several papers (see, for example, [2, 7, 8]). For double Cesàro methods this problem has in recent years been considered, for example, in [6, 9].

In the present paper the particular subcase of the above-described problem is studied: sufficient conditions for a matrix \( M = (m_{nk}) \), defined by the relation

\[
m_{nk} = \begin{cases} 
A_{n-k}^s & (k \leq n, s \in \mathbb{C}), \\
0 & (k > n),
\end{cases}
\]

(1.1)
to be a transform from \( c_{C^\alpha} \) into \( c_{C^\beta} \), \( \alpha, \beta \in \mathbb{C}\{−1, −2, . . .\} \) are found. It is easy to see that this problem is equivalent to the problem of finding sufficient conditions for \( c_{C^\alpha} \subset c_G \), where \( G := C^\beta M \).

2 Auxiliary Results

For the proof of main results we need the following properties of Cesàro numbers (see [4], p. 77–81):

\[
\sum_{k=0}^{n} A_{n-k}^\alpha A_{k}^\beta = A_{n}^{\alpha+\beta+1} \quad \text{for every } \alpha, \beta \in \mathbb{C},
\]

(2.1)

\[
|A_{n}^\alpha| \leq K_1 (n+1)^{Re \alpha} \quad \text{for every } \alpha \in \mathbb{C}, \ K_1 > 0,
\]

(2.2)

\[
|A_{n}^\alpha| \geq K_2 (n+1)^{Re \alpha} \quad \text{for } \alpha \in \mathbb{C}\{−1, −2, . . .\}, \ K_2 > 0.
\]

(2.3)

Further we also use the following lemma.

**Lemma 1.** Let \( \alpha \in \mathbb{C}, \beta \in \mathbb{C} \). The following assertions hold:

(A) If \( Re \alpha \neq −1 \) and \( Re \beta \neq −1, \) or \( \alpha = \beta = −1, \) then

\[
D_n := \sum_{k=0}^{n} |A_{n-k}^\alpha A_{k}^\beta| = O [(n+1)^{Re \alpha} + O [(n+1)^{Re \beta} + O [(n+1)^{Re (\alpha+\beta)+1}].
\]

(2.4)

(B) If \( Re \beta = −1, \) then

\[
D_n = \begin{cases} 
O [(n+1)^{Re \alpha} \ln(n+1)] & (Re \alpha \geq −1), \\
O [(n+1)^{-1}] & (Re \alpha < −1).
\end{cases}
\]

(C) If \( Re \alpha = −1, \) then

\[
D_n = \begin{cases} 
O [(n+1)^{Re \beta} \ln(n+1)] & (Re \beta \geq −1), \\
O [(n+1)^{-1}] & (Re \beta < −1).
\end{cases}
\]

**Proof.** First we note that for all \( \alpha, \beta \in \mathcal{R} \) relation (2.4) is proved, for example, in [4], p. 79–81. Let now \( Re \alpha \neq −1, Re \beta \neq −1. \) Then by (2.2) and (2.3) there
exist $K_1^1, K_1^2, K_2^1, K_2^2 > 0$ so that

$$D_n \leq K_1^1 K_1^2 \sum_{k=0}^{n} (n-k+1)^{Re \alpha (k+1)} Re \beta \leq K_1^1 K_2^1 K_2^2 \sum_{k=0}^{n} |A_{n-k}^{Re \alpha} A_k^{Re \beta}|.$$ 

Hence relation (2.4) holds, since $Re \alpha, Re \beta \in \mathcal{R}$. Thus assertion (A) is satisfied.

Let now $Re \beta = -1$. Then with the help of relation (2.2) we have

$$D_n \leq MV_n, \quad M > 0,$$

where

$$V_n := \sum_{k=0}^{n} v_{nk}, \quad v_{nk} := (n - k + 1)^{Re \alpha (k+1)} - 1.$$ 

Further we can write

$$V_n = \begin{cases} \sum_{k=0}^{n} v_{nk} + \left( \frac{n}{2} + 1 \right)^{Re \alpha - 1} + \sum_{k=\frac{n}{2}+1}^{n} v_{nk} & \text{(n is even)}, \\ \sum_{k=0}^{n} v_{nk} + \sum_{k=\frac{n}{2}+1}^{n} v_{nk} & \text{(n is odd)}, \end{cases}$$

i.e.,

$$V_n = (n + 1)^{Re \alpha - 1} + n^{Re \alpha 2 - 1} + \cdots + \left( \frac{n}{2} + 3 \right)^{Re \alpha \left( \frac{n}{2} - 1 \right)}$$

$$+ \left( \frac{n}{2} + 2 \right)^{Re \alpha \left( \frac{n}{2} \right)} - 1 + \left( \frac{n}{2} + 1 \right)^{Re \alpha - 1} + \left( \frac{n}{2} \right)^{Re \alpha \left( \frac{n}{2} + 2 \right)} - 1$$

$$+ \cdots + 2^{Re \alpha n - 1} + 1^{Re \alpha (n + 1)} - 1$$

for an even number $n$, and

$$V_n = (n + 1)^{Re \alpha - 1} + n^{Re \alpha 2 - 1} + \cdots + \left( \frac{n}{2} + 5 \right)^{Re \alpha \left( \frac{n}{2} - 1 \right)}$$

$$+ \left( \frac{n}{2} + 3 \right)^{Re \alpha \left( \frac{n}{2} \right)} - 1 + \left( \frac{n}{2} + 2 \right)^{Re \alpha \left( \frac{n}{2} + 1 \right)} - 1$$

$$+ \left( \frac{n}{2} + 1 \right)^{Re \alpha \left( \frac{n}{2} + 2 \right)} - 1 + \cdots + 2^{Re \alpha n - 1} + 1^{Re \alpha (n + 1)} - 1$$

for an odd number $n$. Let us suppose first that $Re \alpha \geq -1$. Then

$$(n + 1)^{Re \alpha 1 - 1} \geq 1^{Re \alpha (n + 1)} - 1,$$

$$n^{Re \alpha 2 - 1} \geq 2^{Re \alpha n - 1},$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$\left( \frac{n}{2} + 2 \right)^{Re \alpha \left( \frac{n}{2} \right)} - 1 \geq \left( \frac{n}{2} \right)^{Re \alpha \left( \frac{n}{2} + 2 \right)} - 1,$$
if \( n \) is an even number, and
\[
(n + 1)^{Re \alpha_1 - 1} \geq 1^{Re \alpha}(n + 1)^{-1},
\]
\[
n^{Re \alpha_2 - 1} \geq 2^{Re \alpha} n^{-1},
\]

\[
\left( \frac{n + 3}{2} \right)^{Re \alpha} \left( \frac{n + 1}{2} \right)^{-1} \geq \left( \frac{n + 1}{2} \right)^{Re \alpha} \left( \frac{n + 3}{2} \right)^{-1},
\]

if \( n \) is an odd number. Consequently
\[
V_n \leq \begin{cases} 
2 \sum_{k=0}^{\frac{n-1}{2}} \left( n - k + 1 \right)^{Re \alpha} (k + 1)^{-1} + \left( \frac{n}{2} + 1 \right)^{Re \alpha - 1} & (n \text{ is even}), \\
2 \sum_{k=0}^{\frac{n-1}{2}} \left( n - k + 1 \right)^{Re \alpha} (k + 1)^{-1} & (n \text{ is odd}). 
\end{cases}
\]

Therefore
\[
V_n \leq \begin{cases} 
2 \left( \frac{n}{2} + 2 \right)^{Re \alpha} \sum_{k=0}^{\frac{n-1}{2}} (k + 1)^{-1} + \left( \frac{n}{2} + 1 \right)^{Re \alpha - 1} & (n \text{ is even}), \\
2(n + 1)^{Re \alpha} \sum_{k=0}^{\frac{n-1}{2}} (k + 1)^{-1} & (n \text{ is odd}) 
\end{cases}
\]

if \(-1 \leq Re \alpha \leq 0\), and
\[
V_n \leq \begin{cases} 
2(n + 1)^{Re \alpha} \sum_{k=0}^{\frac{n-1}{2}} (k + 1)^{-1} + \left( \frac{n}{2} + 1 \right)^{Re \alpha - 1} & (n \text{ is even}), \\
2(n + 1)^{Re \alpha} \sum_{k=0}^{\frac{n-1}{2}} (k + 1)^{-1} & (n \text{ is odd}) 
\end{cases}
\]

if \( Re \alpha > 0 \). Hence
\[
D_n = O \left[ (n + 1)^{Re \alpha \ln(n + 1)} \right]; \quad Re \alpha \geq -1. \quad (2.5)
\]

We assume now that \( Re \alpha < -1 \). Then
\[
(n + 1)^{Re \alpha_1 - 1} \leq 1^{Re \alpha}(n + 1)^{-1},
\]
\[
n^{Re \alpha_2 - 1} \leq 2^{Re \alpha} n^{-1},
\]

\[
\left( \frac{n + 3}{2} \right)^{Re \alpha} \left( \frac{n + 1}{2} \right)^{-1} \leq \left( \frac{n + 1}{2} \right)^{Re \alpha} \left( \frac{n + 3}{2} \right)^{-1},
\]

if \( n \) is an even number, and
\[
(n + 1)^{Re \alpha_1 - 1} \leq 1^{Re \alpha}(n + 1)^{-1},
\]
\[
n^{Re \alpha_2 - 1} \leq 2^{Re \alpha} n^{-1},
\]

\[
\left( \frac{n + 3}{2} \right)^{Re \alpha} \left( \frac{n + 1}{2} \right)^{-1} \leq \left( \frac{n + 1}{2} \right)^{Re \alpha} \left( \frac{n + 3}{2} \right)^{-1},
\]
if $n$ is an odd number. Consequently
\[
V_n \leq \left\{ \begin{array}{ll}
2 \sum_{k=\frac{n+1}{2}}^{n} (n - k + 1)^{Re \alpha} (k + 1)^{-1} + \left( \frac{n}{2} + 1 \right)^{Re \alpha - 1} & (n \text{ is even}), \\
2 \sum_{k=\frac{n+1}{2}}^{n} (n - k + 1)^{Re \alpha} (k + 1)^{-1} & (n \text{ is odd}).
\end{array} \right.
\]
Therefore
\[
V_n \leq \left\{ \begin{array}{ll}
4(n + 4)^{-1} \sum_{k=\frac{n+1}{2}}^{n} (n - k + 1)^{Re \alpha} + \left( \frac{n}{2} + 1 \right)^{Re \alpha - 1} & (n \text{ is even}), \\
4(n + 3)^{-1} \sum_{k=\frac{n+1}{2}}^{n} (n - k + 1)^{Re \alpha} & (n \text{ is odd}).
\end{array} \right.
\]
Hence
\[
D_n = O \left[ (n + 1)^{-1} \right]; \quad Re \alpha < -1.
\] (2.6)
Consequently assertion (B) holds by (2.5) and (2.6). The proof of assertion (C) is similar to the proof of assertion (B). So we omit it. $\Box$

3 Main Results

Now we are able to prove the main result of this paper.

**Theorem 1.** Let $\alpha, \beta \in \mathbb{C}\setminus\{-1, -2, \ldots\}$ and $s \in \mathbb{C}$. If $Re s < -1$ and $Re s < Re \alpha \leq Re \beta$, then the matrix $M = (m_{nk})$, defined by relation (1.1), transforms $c_{C^\alpha}$ into $c_{C^\beta}$.

**Proof.** It is sufficient to show that $c_{C^\alpha} \subset c_G$, where $G = C^\beta M := (g_{nk})$.

Using equality (2.1), we get
\[
g_{nl} = \frac{1}{A_n^\beta} \sum_{k=l}^{n} A_{n-k}^\beta A_{k-l}^s = \frac{1}{A_n^\beta} \sum_{k=0}^{n-l} A_{n-l-k}^\beta A_k^s = \frac{A_{n-l}^{\beta + s + 1}}{A_n^\beta}.
\]
As the inverse matrix $(\eta_{lk})$ of $C^\alpha = (c_{nk})$ is defined by the equalities (see [4], p. 86)
\[
\eta_{lk} = \left\{ \begin{array}{ll}
A_k^\alpha A_{l-k}^{-\alpha - 2} & (k \leq l), \\
0 & (k > l),
\end{array} \right.
\]
for every $x = (x_k) \in c_{C^\alpha}$ we get
\[
\sum_{k=0}^{n} g_{nk} x_k = \sum_{k=0}^{n} \gamma_{nk} y_k,
\]
where $y_k := C_k^\alpha x$ and $\gamma_{nk} = A_{n-k}^{\beta + s - \alpha} A_k^\alpha / A_n^\beta$ by equality (2.1). Consequently, for $c_{C^\alpha} \subset c_G$ it is sufficient to show by the well-known theorem of Kojima-Schur that
\[
there \ exists \ the \ finite \ limits \ \lim_n \gamma_{nk}, \ \lim_n \sum_{k=0}^{n} \gamma_{nk}, \quad (3.1)
\]
and
\[ \sum_{k} |\gamma_{nk}| = O(1), \] (3.2)

since the sequence \((y_k)\) is convergent for every \(x \in c_{C^\alpha}\). Thus, with the help of relations (2.1)--(2.3) we have
\[ \left| \sum_{k=0}^{n} \gamma_{nk} \right| = \left| \frac{A_n^{\beta+s+1}}{A_n^\beta} \right| = O(1) (n+1)^{Re s+1} = o(1) \]
(since \(Re s + 1 < 0\)), and
\[ |\gamma_{nk}| = O(1) \left( \frac{(n-k+1)^{Re(\beta+s-\alpha)}}{(n+1)^{Re \beta}} \right) \]
\[ = O(1) \left( 1 - \frac{k}{n+1} \right)^{Re \beta} (n-k+1)^{Re(s-\alpha)} = o(1) \]
(since \(Re (s - \alpha) < 0\)). Thus condition (3.1) is fulfilled.

The proof of validity of condition (3.2) we divide into three parts.
1) Let \(Re (\beta + s - \alpha) \neq -1\), \(Re \alpha \neq -1\), or \(\beta + s - \alpha = \alpha = -1\). Then we get
\[ S_n := \sum_{k=0}^{n} \left| A_{n-k}^{\beta+s-\alpha} A_k^\alpha \right| = O \left[ (n+1)^{Re (\beta+s-\alpha)} \right] + O \left[ (n+1)^{Re \alpha} \right] \]
\[ + O \left[ (n+1)^{Re (\beta+s)+1} \right] \]
by Lemma 1. If
\[ L := \max \{ Re (\beta + s - \alpha), Re \alpha, Re (\beta + s) + 1 \} = Re (\beta + s - \alpha), \]
then \(S_n = O \left[ (n+1)^{Re (\beta+s-\alpha)} \right] \), and consequently with the help of (2.3) we have
\[ T_n := \sum_{k=0}^{n} |\gamma_{nk}| = \frac{S_n}{|A_n^\beta|} = O \left[ (n+1)^{Re (s-\alpha)} \right] = O(1). \]
If \(L = Re (\beta + s) + 1\), then using (2.3) we can conclude that
\[ S_n = O \left[ (n+1)^{Re (\beta+s)+1} \right], \]
and therefore
\[ T_n = O \left[ (n+1)^{Re s+1} \right] = O(1). \]
If \(L = Re \alpha\), then \(S_n = O \left[ (n+1)^{Re \alpha} \right] \), and hence
\[ T_n = O \left[ (n+1)^{Re (\alpha-\beta)} \right] = O(1), \]
i.e. condition (3.2) holds.
2) Let $Re \alpha = -1$. Then

$$S_n = \begin{cases} O \left[ (n+1)^{Re(\beta+s-\alpha)} \ln(n+1) \right] & (Re(\beta+s-\alpha) \geq -1), \\ O \left[ (n+1)^{-1} \right] & (Re(\beta+s-\alpha) < -1) \end{cases}$$

and consequently

$$T_n = \begin{cases} O \left[ (n+1)^{Re(s-\alpha)} \ln(n+1) \right] & (Re(\beta+s-\alpha) \geq -1), \\ O \left[ (n+1)^{-Re\beta-1} \right] & (Re(\beta+s-\alpha) < -1) \end{cases}$$

Therefore $T_n = O(1)$, because $Re(s-\alpha) < 0$ and $Re \beta \geq Re \alpha = -1$, i.e. condition (3.2) holds.

3) Let $Re(\beta+s-\alpha) = -1$. Then

$$S_n = \begin{cases} O \left[ (n+1)^{Re\alpha} \ln(n+1) \right] & (Re\alpha \geq -1), \\ O \left[ (n+1)^{-1} \right] & (Re\alpha < -1) \end{cases}$$

and consequently

$$T_n = \begin{cases} O \left[ (n+1)^{Re(\alpha-\beta)} \ln(n+1) \right] & (Re\alpha \geq -1), \\ O \left[ (n+1)^{-Re\beta-1} \right] & (Re\alpha < -1) \end{cases}$$

Therefore $T_n = O(1)$, because $Re(\alpha-\beta) = Re s + 1 < 0$ and $-Re \beta - 1 = Re(s-\alpha) < 0$, i.e. condition (3.2) holds. □

It is well known that $C^\beta$ includes $C^\alpha$, i.e. $c_{C^\beta} \supseteq c_{C^\alpha}$, if $Re \beta > Re \alpha > -1$ (see [4], p. 87). Therefore for real numbers $\alpha, \beta$ we get

Corollary 1. Let $\alpha, \beta \in \mathcal{R}, s \in \mathcal{C}$ with $\alpha, \beta > -1$, $Re s < -1$ and $M$ be defined by (1.1). If $M$ transforms $c_{C^\alpha}$ into $c_{C^\beta}$, then $C^\beta$ includes $C^\alpha$.

Proof. We see from the proof of Theorem 1 that the validity of condition (3.2) is necessary for $M$ to be a transformation from $c_{C^\alpha}$ into $c_{C^\beta}$. We prove that condition (3.2) is not satisfied for $\beta < \alpha$. Indeed, by (2.2) and (2.3) there exists a number $K > 0$ so that

$$T_n = \sum_{k=0}^{n} \frac{A_{n+k}^{\beta+s-\alpha} A_{k}^{\alpha}}{A_{n}^{\beta}} = \sum_{k=0}^{n-1} \left| \frac{A_{n+k}^{\beta+s-\alpha} A_{k}^{\alpha}}{A_{n+k}^{\beta}} \right| + \left| \frac{A_{n}^{\alpha}}{A_{n}^{\beta}} \right| \geq K \left[ (n+1)^{Re(\alpha-\beta)} \right].$$

As the sequence $((n+1)^{Re(\alpha-\beta)})$ is not bounded for $\beta < \alpha$, then also the sequence $(T_n)$ is not bounded for $\beta < \alpha$. Consequently for the validity of condition (3.2) it is necessary that $\beta \geq \alpha$. As $\alpha, \beta > -1$, then $C^\beta$ includes $C^\alpha$. □

References


