Numerical Solution of Volterra Integral Equations with Weakly Singular Kernels which May Have a Boundary Singularity∗

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Abstract. We propose a piecewise polynomial collocation method for solving linear Volterra integral equations of the second kind with kernels which, in addition to a weak diagonal singularity, may have a weak boundary singularity. Global convergence estimates are derived and a collection of numerical results is given.

Key words: Volterra integral equation, weakly singular kernel, boundary singularity, collocation method.

1 Introduction

Let $C^k(\Omega)$ be the set of all $k$ times continuously differentiable functions on $\Omega$, $C^0(\Omega) = C(\Omega)$. Let $b \in \mathbb{R} = (-\infty, \infty)$, $b > 0$,

$$D_b = \{(x, y) : 0 \leq x \leq b, \ 0 < y < x\}, \quad \overline{D}_b = \{(x, y) : 0 \leq y \leq x \leq b\}.$$ 

In many practical applications (see, for example, [3, 5]) there arise integral equations of the form

$$u(x) = \int_0^x K(x, y)u(y)dy + f(x), \quad 0 \leq x \leq b, \tag{1.1}$$

with $f \in C^m[0, b]$, $K(x, y) = g(x, y)(x - y)^{-\nu}$, $0 < \nu < 1$, $g \in C^m(\overline{D}_b)$, $m \in \mathbb{N} = \{1, 2, \ldots\}$. The solution $u(x)$ to (1.1) is typically non-smooth at $x = 0$ where its derivatives become unbounded (see, for example, [3, 4, 5, 9]). In collocation methods the singular behaviour of the solution $u(x)$ can be taken into account by using polynomial splines on special graded grids

$$\Delta^r_N = \{x_0, \ldots, x_N : 0 = x_0 < \ldots < x_N = b\}$$

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The parameter $r$ characterizes the degree of non-uniformity of the grid $\Delta_N$: if $r > 1$, then the nodes $x_0, \ldots, x_N$ of the grid $\Delta_N$ are more densely clustered near the left endpoint of the interval $[0, b]$ where $u(x)$ may be singular. By using a collocation method based on the grid $\Delta_N$ and piecewise polynomials of degree at most $m - 1$ one can reach a convergence of order $O(N^{-m})$ for $r \geq m/(1 - \nu)$, see [3, 4, 5]. However, although the piecewise polynomial collocation method on $\Delta_N$ turns out to be stable for solving weakly singular integral equations (see [8]), the realization of this method in case of strongly graded grids $\Delta_N$ by large values of $r$ may lead to unstable behaviour of numerical results.

To avoid problems associated with the use of strongly graded grids the following approach for solving (1.1) can be used: first we perform in (1.1) a change of variables so that the singularities of the derivatives of the solution will be milder or disappear and after that we solve the transformed equation by a collocation method on a mildly graded or uniform grid. We refer to [13] for details (see also [2, 7, 12]). Note that in [10, 15] similar ideas for solving Fredholm integral equations have been used (see also [6, 11, 16]).

In the present paper we extend the domain of applicability of this approach. To this aim, we examine a more complicated situation for equation (1.1) where the kernel $K(x, y)$, in addition to a diagonal singularity (a singularity as $y \to x$), may have a boundary singularity (a singularity as $y \to 0$). Actually, we assume that the kernel $K(x, y)$ has the form

\[ K(x, y) = g(x, y)(x - y)^{-\nu}y^{-\lambda}, \quad (x, y) \in D_b, \quad 0 < \nu < 1, \quad 0 \leq \lambda < 1, \quad (1.3) \]

where $g \in C^m(\overline{D_b})$, $m \in \{0\} \cup \mathbb{N}$. The set of kernels satisfying (1.3) will be denoted by $W^{m, \nu, \lambda}(D_b)$.

Throughout the paper $c$ denotes a positive constant which may have different values by different occurrences.

### 2 Regularity of the Solution

For given $m \in \mathbb{N}$ and $0 < \theta < 1$ let $C^m, \theta(0, b]$ be the set of functions $u \in C[0, b] \cap C^m(0, b]$ such that

\[ |u^{(j)}(x)| \leq cx^{1-\theta-j}, \quad 0 < x \leq b, \quad j = 1, \ldots, m. \quad (2.1) \]

It follows from [14] that the regularity of the solution to (1.1) can be characterized by the following result.

**Lemma 1.** Assume that $K \in W^{m, \nu, \lambda}(D_b)$ and $f \in C^{m, \nu+\lambda}(0, b]$ where $m \in \mathbb{N}$, $0 < \nu < 1$, $0 \leq \lambda < 1$, $\nu + \lambda < 1$. Then equation (1.1) has a unique solution $u \in C^{m, \nu+\lambda}(0, b]$. 

3 Smoothing Transformation

For given $\varphi \in [1, \infty)$ denote
\[ \varphi(s) = b^{1-e} s^e, \quad 0 \leq s \leq b. \] (3.1)

Clearly, $\varphi \in C[0, b]$, $\varphi(0) = 0$, $\varphi(b) = b$ and $\varphi'(s) > 0$ for $0 < s \leq b$. Thus, $\varphi$ maps $[0, b]$ onto $[0, b]$ and has a continuous inverse $\varphi^{-1} : [0, b] \rightarrow [0, b]$,
\[ \varphi^{-1}(x) = b^{(e-1)/e} x^{1/e}, \quad 0 \leq x \leq b. \]

Note that $\varphi(s) \equiv s$ for $g = 1$. We are interested in a transformation (3.1) with $g > 1$ since it possesses a smoothing property for $u(\varphi(s))$ with singularities of $u'(x), \ldots, u^{(m)}(x)$ at $x = 0$ (see Lemma 2).

**Lemma 2.** Let $u \in C^{m,\theta}(0, b)$, $m \in \mathbb{N}$, $0 < \theta < 1$, and let $\varphi$ be the transformation (3.1). Furthermore, let
\[ u_\varphi(s) = u(\varphi(s)), \quad 0 \leq s \leq b. \]

Then $u_\varphi \in C[0, b] \cap C^m(0, b)$ and
\[ |u^{(j)}_\varphi(s)| \leq c s \theta^{(1-\theta) - j}, \quad 0 < s \leq b, \quad j = 1, \ldots, m. \] (3.2)

**Proof.** The smoothness claim is clear. Further, for the derivatives of the composite function $u_\varphi = u \circ \varphi$, we have the Faà di Bruno’s representation
\[ u^{(j)}_\varphi(s) = \sum \frac{j!}{n_1! \cdots n_j!} u^{(n)}(\varphi(s)) \left( \frac{\varphi'(s)}{1!} \right)^{n_1} \cdots \left( \frac{\varphi^{(j)}(s)}{j!} \right)^{n_j}, \] (3.3)
where $0 < s \leq b$, $n = n_1 + \ldots + n_j$ and the sum is taken over all $n_1, \ldots, n_j \in \{0\} \cup \mathbb{N}$ for which $n_1 + 2 n_2 + \ldots + j n_j = j$, $j = 1, \ldots, m$. It follows from (2.1), (3.1), $n = n_1 + \ldots + n_j$ and $n_1 + 2 n_2 + \ldots + j n_j = j$ that
\[ \left| u^{(n)}(\varphi(s)) (\varphi'(s))^{n_1} \cdots (\varphi^{(j)}(s))^{n_j} \right| \leq c s \theta^{(1-\theta) - j}, \quad 0 < s \leq b. \]

This together with (3.3) yields (3.2). $\square$

**Remark 1.** Instead of (3.1) other transformations are possible. We refer to [13] for a general discussion in this connection.

4 Numerical Method

Using (3.1) we introduce in (1.1) the change of variables $y = \varphi(s)$, $x = \varphi(t)$, $s, t \in [0, b]$. We obtain an integral equation of the form
\[ u_\varphi(t) = \int_0^t K_\varphi(t, s) u_\varphi(s) ds + f_\varphi(t), \quad 0 \leq t \leq b, \] (4.1)

where
\[ f_\varphi(t) = f(\varphi(t)), \quad K_\varphi(t, s) = K(\varphi(t), \varphi(s)) \varphi'(s) \]
are given functions and \( u_\varphi(t) = u(\varphi(t)) \) is a function which we have to find.

For given integers \( m, N \in \mathbb{N} \) let

\[
S_{m-1}(\Delta_N^r) = \left\{ v_N : v_N|_{[x_{j-1}, x_j]} \in \pi_{m-1}, j = 1, \ldots, N \right\},
\]

\[
S_{m-1}(\Delta_N^r) = \left\{ v_N \in C[0, b] : v_N|_{[x_{j-1}, x_j]} \in \pi_{m-1}, j = 1, \ldots, N \right\}
\]

be the underlying spline spaces of piecewise polynomial functions on the grid \( \Delta_N^r \) with the nodes (1.2). Here \( v_N|_{[x_{j-1}, x_j]} (j = 1, \ldots, N) \) is the restriction of \( v_N(t), t \in [0, b] \), to the subinterval \([x_{j-1}, x_j] \subset [0, b]\) and \( \pi_{m-1} \) denotes the set of polynomials of degree not exceeding \( m - 1 \). Note that the elements of \( S_{m-1}(\Delta_N^r) \) may have jump discontinuities at the interior knots \( x_1, \ldots, x_{N-1} \) of the grid \( \Delta_N^r \). In every subinterval \([x_{j-1}, x_j] (j = 1, \ldots, N) \) we introduce \( m \in \mathbb{N} \) interpolation (collocation) points

\[
x_{jl} = x_{j-1} + \eta_l (x_j - x_{j-1}), \quad l = 1, \ldots, m; \; j = 1, \ldots, N,
\]

(4.2)

where \( \eta_1, \ldots, \eta_m \) are some fixed (collocation) parameters such that

\[
0 \leq \eta_1 < \ldots < \eta_m \leq 1.
\]

We find an approximation \( v_N = v_{N,m,r,\varphi} \) to \( u_\varphi \), the solution of equation (4.1) (under the conditions of Theorem 1 below the equations (1.1) and (4.1) are uniquely solvable), by collocation method from the following conditions:

\[
v_N \in S_{m-1}^{(-1)}(\Delta_N^r), \quad N, m \in \mathbb{N}, \; r \geq 1,
\]

(4.4)

\[
v_N(x_{jl}) = \int_0^{x_{jl}} K_\varphi(x_{jl}, s)v_N(s) \, ds + f_\varphi(x_{jl}), \quad l = 1, \ldots, m; \; j = 1, \ldots, N,
\]

(4.5)

With \( x_{jl}, l = 1, \ldots, m; \; j = 1, \ldots, N \), given by formula (4.2).

Having determined the approximation \( v_N \) for \( u_\varphi \), we determine an approximation \( u_N = u_{N,m,r,\varphi} \) for \( u \), the solution of equation (1.1), setting

\[
u_N(x) = v_N(\varphi^{-1}(x)), \quad 0 \leq x \leq b.
\]

(4.6)

**Remark 2.** The choice of nodes (4.2) with \( \eta_1 = 0, \eta_m = 1 \) in (4.5) actually implies that the resulting collocation approximation \( v_N \) belongs to the smoother spline space \( S_{m-1}^{(0)}(\Delta_N^r) \) than it is stated by the condition (4.4).

**Remark 3.** The settings (4.4), (4.5) form a linear system of algebraic equations whose exact form is determined by the choice of a basis in \( S_{m-1}^{(-1)}(\Delta_N^r) \). We refer to [13] for a convenient choice of it.

## 5 Convergence Results

Let \( X \) and \( Y \) be Banach spaces. By \( \mathcal{L}(X,Y) \) we denote the Banach space of all linear continuous operators \( A : X \to Y \) with the norm

\[
\|A\|_{\mathcal{L}(X,Y)} = \sup\{\|Az\| : z \in X, \; \|z\|_X \leq 1\}.
\]

By \( C[a, b] \) we denote the Banach space of continuous functions \( z \) on \([a, b]\) with the usual norm \( \|z\| = \max\{|z(t)| : t \in [a, b]\} \).
**Theorem 1.** Let \( f \in C[0,b] \) and \( K \in W^{0,\nu,\lambda}(D_b) \), \( 0 < \nu < 1, 0 \leq \lambda < 1 - \nu \). Furthermore, assume that \( \varphi \) is the transformation (3.1) and the interpolation nodes (4.2) with grid points (1.2) and parameters (4.3) are used. Then equation (1.1) has a unique solution \( u \in C[0,b] \), the settings (4.4)–(4.6) determine for sufficiently large \( N \) a unique approximation \( u_N \) for \( u \) and

\[
\|u_N - u\|_{\infty} \to 0 \quad \text{as} \quad N \to \infty, \tag{5.1}
\]

where \( \|u_N - u\|_{\infty} = \sup_{0 \leq x \leq b} |u_N(x) - u(x)| \).

**Proof.** We write (4.1) in the form \( u_\varphi = T_\varphi u_\varphi + f_\varphi \) where \( T_\varphi \) is defined by formula

\[
(T_\varphi z)(t) = \int_0^t K_\varphi(t, s)z(s) \, ds, \quad 0 \leq t \leq b.
\]

It follows from (1.3) and (3.1) that \( K_\varphi(t, s) \) is continuous in \( D_b \) and

\[
|K_\varphi(t, s)| \leq c(t - s)^{-\nu}s^{-\lambda}, \quad (t, s) \in D_b.
\]

Since \( \nu + \lambda < 1 \), \( T_\varphi \) is compact as an operator from \( L^\infty(0, b) \) into \( C[0, b] \), see [14]. This together with \( f_\varphi \in C[0, b] \) yields that equation \( u_\varphi = T_\varphi u_\varphi + f_\varphi \) (equation (4.1)) has a unique solution \( u_\varphi \in C[0, b] \). In particular, (1.1) has a unique solution \( u \in C[0, b] \).

Further, conditions (4.4), (4.5) have the operator equation representation

\[
v_N = P_N T_\varphi v_N + P_N f_\varphi, \tag{5.2}
\]

where \( P_N \) is an operator which assigns to every continuous function \( z \in C[0, b] \) its piecewise polynomial function \( P_N z \in S^{(m-1)}(\Delta_N^r) \) such that \( (P_N z)(x_{jl}) = z(x_{jl}), \ l = 1, \ldots, m; \ j = 1, \ldots, N \). It follows from [17] that the norms of \( P_N \in \mathcal{L}(C[0, b], L^\infty(0, b)) \) are bounded by a constant \( c \) which is independent of \( N \),

\[
\|P_N\|_{\mathcal{L}(C[0, b], L^\infty(0, b))} \leq c, \tag{5.3}
\]

and

\[
\|z - P_N z\|_{\infty} \to 0 \quad \text{as} \quad N \to \infty \quad \text{for every} \quad z \in C[0, b]. \tag{5.4}
\]

Using a standard argumentation (cf. [13, 15, 17]) we obtain that equation (5.2) has for sufficiently large values of \( N \), say \( N \geq N_0 \), a unique solution \( v_N \in S^{(m-1)}(\Delta_N^r) \) and

\[
\|v_N - u_\varphi\|_{\infty} \leq c\|u_\varphi - P_N u_\varphi\|_{\infty}, \quad N \geq N_0. \tag{5.5}
\]

Here \( u_\varphi \) is the solution of equation (4.1) and \( c \) is a positive constant not depending on \( N \). Since \( u_\varphi \in C[0, b] \), we get from (5.4) and (5.5) that \( \|v_N - u_\varphi\|_{\infty} \to 0 \) as \( N \to \infty \). This together with

\[
\|u_N - u\|_{\infty} = \|v_N - u_\varphi\|_{\infty} \tag{5.6}
\]

yields (5.1). \( \Box \)

Next we establish a global convergence result for method (4.4)–(4.6).
Theorem 2. Let the following conditions be fulfilled:

1. $K \in W^{m,\nu,\lambda}(D_b)$, $f \in C^{m,\nu+\lambda}(0, b]$, $m \in \mathbb{N}$, $0 < \nu < 1$, $0 \leq \lambda < 1 - \nu$;
2. $\varphi$ is the transformation (3.1);
3. the interpolation nodes (4.2) with grid points (1.2) and parameters (4.3) are used.

Then the settings (4.4)–(4.6) determine for $N \geq N_0$ a unique approximation $u_N$ to $u$, the solution to (1.1), and

$$
\|u_N - u\|_{\infty} \leq c \left\{ \begin{array}{ll}
N^{-r(1-\nu-\lambda)} & \text{for } 1 \leq r < \frac{m}{e(1-\nu-\lambda)}, \\
N^{-m} & \text{for } r \geq \frac{m}{e(1-\nu-\lambda)}, \ r \geq 1,
\end{array} \right.
$$

(5.7)

where $c$ is a positive constant not depending on $N$.

Proof. On the basis of Lemmas 1 and 2 we find that $u_\varphi \in C[0, b] \cap C^m(0, b]$ and for every $s \in (0, b]$ and $j = 1, \ldots, m$,

$$
|u_\varphi^{(j)}(s)| \leq c \left\{ \begin{array}{ll}
1 & \text{if } j \leq g(1 - \nu - \lambda), \\
se^{(1-\nu-\lambda)j} & \text{if } j > g(1 - \nu - \lambda).
\end{array} \right.
$$

(5.8)

For a spline $w_N \in S_m^{-1}(\Delta_N)$ denote $w_{N, j} = w_N|_{x_{j-1}, x_j}$, $j = 1, \ldots, N$. Due to (5.3) we get the estimate

$$
\|u_\varphi - P_N u_\varphi\|_{\infty} = \|u_\varphi - w_N - P_N(\varphi - w_N)\|_{\infty} \\
\leq c \max_{j=1, \ldots, N} \max_{x_{j-1} \leq x \leq x_j} |u_\varphi(x) - w_{N, j}(x)|,
$$

(5.9)

with a positive constant $c$ which is independent of $N$. We fix $w_{N, j}$ as a Taylor polynomial for $u_\varphi(x)$ at $x = x_j$:

$$
w_{N, j}(x) = \sum_{k=0}^{m-1} \frac{u_\varphi^{(k)}(x_j)}{k!} (x - x_j)^k, \quad x_{j-1} \leq x \leq x_j.
$$

The integral form of the reminder term of the $(m - 1)$th order Taylor approximation of $u_\varphi(x)$ at $x = x_j$ and the estimate (5.8) gives us for all $x \in [x_{j-1}, x_j]$ ($j = 1, \ldots, N$) the inequality

$$
|u_\varphi(x) - w_{N, j}(x)| \leq c \int_x^{x_j} (s-x)^{m-1} \left\{ \begin{array}{ll}
1 & \text{if } m \leq g(1 - \nu - \lambda), \\
se^{(1-\nu-\lambda)m} & \text{if } m > g(1 - \nu - \lambda)
\end{array} \right. ds.
$$

(5.10)

Due to (1.2),

$$
x_j - x_{j-1} \leq brN^{-1}, \quad j = 1, \ldots, N.
$$

(5.11)

If $m \leq g(1 - \nu - \lambda)$, then we obtain from (5.10) and (5.11) that

$$
|u_\varphi(x) - w_{N, j}(x)| \leq cN^{-m}, \quad x_{j-1} \leq x \leq x_j, \quad j = 1, \ldots, N.
$$

(5.12)
where $c$ is a positive constant not depending on $N$.

In the case $m > q(1 - \nu - \lambda)$ we have

$$\max_{0 \leq x \leq x_1} \int_x^{x_1} (s-x)^{m-1} s^{(1-\nu-\lambda)-m} ds \leq \max_{0 \leq x \leq x_1} \int_x^{x_1} (s-x)^{(1-\nu-\lambda)-1} ds$$

$$\leq c_1 \left\{ \begin{array}{ll} N^{-\nu r (1-\nu-\lambda)} & \text{for } 1 \leq r < \frac{m}{q(1-\nu-\lambda)}, \\
N^{-m} & \text{for } r \geq \frac{m}{q(1-\nu-\lambda)}, \ r \geq 1, \end{array} \right. \quad (5.13)$$

$$\max_{j=2,\ldots,N} \max_{x_{j-1} \leq x \leq x_j} \int_x^{x_j} (s-x)^{m-1} s^{(1-\nu-\lambda)-m} ds$$

$$\leq \max_{j=2,\ldots,N} \max_{x_{j-1} \leq x \leq x_j} x^{(1-\nu-\lambda)-m} \int_x^{x_j} (s-x)^{m-1} ds$$

$$\leq c_2 \left\{ \begin{array}{ll} N^{-\nu r (1-\nu-\lambda)} & \text{for } 1 \leq r < \frac{m}{q(1-\nu-\lambda)}, \\
N^{-m} & \text{for } r \geq \frac{m}{q(1-\nu-\lambda)}, \ r \geq 1, \end{array} \right. \quad (5.14)$$

where $c_1$ and $c_2$ are some positive constants not depending on $N$. It follows from (5.9), (5.10) and (5.12)–(5.14) that

$$\|u - P_N u\|_{1,N} \leq c \left\{ \begin{array}{ll} N^{\nu r (1-\nu-\lambda)} & \text{for } 1 \leq r < \frac{m}{q(1-\nu-\lambda)}, \\
N^{-m} & \text{for } r \geq \frac{m}{q(1-\nu-\lambda)}, \ r \geq 1, \end{array} \right.$$ 

with a positive constant $c$ which is independent of $N$. This together with (5.5) and (5.6) yields (5.7). \qed

**Remark 4.** It follows from Theorem 2 that the accuracy $\|u_N - u\|_{1,N} \leq cN^{-m}$ can be achieved on a mildly graded or uniform grid. As an example, if we assume that $\nu = 2/5$, $\lambda = 1/5$, $m = 3$ (the case of piecewise quadratic polynomials), $\vartheta \geq 15/2$, the maximal convergence order $\|u_N - u\|_{1,N} \leq cN^{-3}$ is available for $r \geq 1$. In particular, the uniform grid with nodes (1.2), $r = 1$, may be used.

**Remark 5.** In addition to Theorem 2, assuming some additional smoothness of $f$ and $g$ (see (1.3)) and choosing more carefully the collocation parameters (4.3), the superconvergence of $u_N$ at the collocation points (4.2) can be established, cf. [1, 3, 4, 5, 13, 17]. More precisely, let $K \in W^{m+1, \nu, \lambda}(D_0)$, $f \in C^{m+1, \nu+\lambda}(0, b)$, $m \in \mathbb{N}$, $0 < \nu < 1$, $0 \leq \lambda < 1 - \nu$, and let the interpolation nodes (4.2) be generated by the grid points (1.2) and by the node points $\eta_1, \ldots, \eta_m$ of a quadrature approximation

$$\int_0^1 z(s) ds \approx \sum_{i=1}^m w_i z(\eta_i), \quad 0 \leq \eta_1 < \ldots < \eta_m \leq 1, \quad (5.15)$$

which, with appropriate weights $\{w_i\}$, is exact for all polynomials of degree $m$.

Then it turns out that for sufficiently large $N$,
\[
\max_{i,j=1,\ldots,N} |u_N(f(x_{ij})) - u(f(x_{ij}))| = \max_{i,j=1,\ldots,N} |v_N(x_{ij}) - u_\varphi(x_{ij})| \\
\leq c \left\{ \begin{array}{ll}
N^{-2(1-\nu-\lambda)} & \text{for } 1 \leq r \leq \frac{m+1-\nu}{2(1-\nu-\lambda)}, \\
N^{-m-(1-\nu)} & \text{for } r \geq \frac{m+1-\nu}{2(1-\nu-\lambda)}, \quad r \geq 1.
\end{array} \right.
\] (5.16)

We will investigate this question in a forthcoming paper where a more general class of integral equations with diagonal and boundary singularities will be discussed.

6 Numerical Example

Let us consider the following equation:
\[
u(x) = \int_0^x (x-y)^{-\nu} y^{-\lambda} u(y) \, dy + f(x), \quad 0 \leq x \leq 1,
\] (6.1)

where $0 < \nu < 1$, $0 \leq \lambda < 1$, $\nu + \lambda < 1$. The forcing function $f$ is selected so that $u(x) = x^{1-\nu-\lambda}$ is the exact solution to (6.1). Actually, this is a problem of the form (1.1), (1.3) where $b = 1$, $g(x,y) \equiv 1$, $K(x,y) = (x-y)^{-\nu} y^{-\lambda}$,
\[
f(x) = x^{1-\nu-\lambda} - x^{2(1-\nu-\lambda)} \frac{\Gamma(1-\nu) \Gamma(2(1-\lambda) - \nu)}{\Gamma(3-2(\nu + \lambda))}, \quad 0 \leq x \leq 1,
\]
\[
\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} \, ds, \quad t > 0.
\]

It is easy to check that in this case $K \in W^{m,\nu,\lambda}(D_1)$ and $f \in C^{m,\nu+\lambda}(0,1]$ for arbitrary $m \in \mathbb{N}$.

Equation (6.1) was solved numerically by method (4.4)–(4.6) for $\nu = 2/5$, $\lambda = 1/5$, $m = 3$, $\eta_1 = (5 - \sqrt{15})/10$, $\eta_2 = 1/2$, $\eta_3 = (5 + \sqrt{15})/10$. Here $\eta_1, \eta_2, \eta_3$ are the node points of the Gauss-Legendre quadrature rule (5.15) by $m = 3$. This formula is exact for all polynomials of degree not exceeding $2m - 1 = 5$.

In Tables 1 and 2 some results for different values of the parameters $N$, $\varphi$ and $r$ are presented. The quantities $\epsilon_N^{(\varphi,r)}$ in Table 1 are approximate values of the norm $\|u_N - u\|_{\infty}$, calculated as follows:
\[
\epsilon_N^{(\varphi,r)} = \max_{l=0,\ldots,10, j=1,\ldots,N} |u_N((\tau_{jl})^{(r)})^{\varphi}) - u((\tau_{jl})^{(r)})^{\varphi})|,
\]

where $\tau_{jl}^{(r)} = x_{j-1} + l(x_j - x_{j-1})/10, \quad l = 0,\ldots,10; \quad j = 1,\ldots,N$, with the grid points $x_j$, defined by formula (1.2) for $b = 1$.

Table 2 shows the dependence of
\[
\gamma_N^{(\varphi,r)} = \max_{l=1,\ldots,m, j=1,\ldots,N} |u_N(\varphi(x_{lj})) - u(\varphi(x_{lj}))| = \max_{l=1,\ldots,m, j=1,\ldots,N} |v_N(x_{lj}) - u_\varphi(x_{lj})|
\]
Numerical Solution of Volterra Integral Equations

on the parameters $N$, $\eta$ and $r$ (see (5.16)). The ratios $\delta_N^{(\rho,r)} = \xi_N^{(\rho,r)} / \xi_N^{(\rho,r)}$, $\tilde{\delta}_N^{(\rho,r)} = \gamma_N^{(\rho,r)} / \gamma_N^{(\rho,r)}$, characterizing the observed convergence rate, are also presented. From Theorem 2 it follows that for sufficiently large $N$,

$$\xi_N^{(\rho,r)} \approx \| u_N - u \|_{\infty} \leq c \begin{cases} N^{-2\rho r/5} & \text{if } 1 \leq \rho r < 15/2, \\
N^{-3} & \text{if } \rho r \geq 15/2. \end{cases} \quad (6.2)$$

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<tr>
<th>$N$</th>
<th>$\xi_N^{(1,1)}$</th>
<th>$\xi_N^{(3,1)}$</th>
<th>$\xi_N^{(7/2,3/2)}$</th>
<th>$\xi_N^{(15/2,1)}$</th>
<th>$\xi_N^{(15/4,2)}$</th>
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<td>$\delta_N^{(3,1)}$</td>
<td>$\delta_N^{(7/2,3/2)}$</td>
<td>$\delta_N^{(15/2,1)}$</td>
<td>$\delta_N^{(15/4,2)}$</td>
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<td>3.1 E - 4</td>
<td>1.7 E - 5</td>
<td>1.8 E - 6</td>
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<tr>
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<td>2.30</td>
<td>4.29</td>
<td>8.14</td>
<td>8.00</td>
</tr>
</tbody>
</table>

Table 1. ($m = 3, \nu = \frac{2}{5}, \lambda = \frac{1}{2}, \eta_1 = \frac{5 - \sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5 + \sqrt{15}}{10}$)

Due to (6.2), the ratio $\delta_N^{(\rho,r)}$ ought to be approximately

$$(N/2)^{-2\rho r/5} / N^{-2\rho r/5} = 2^{2\rho r/5}$$

for $1 \leq \rho r < \frac{15}{2}$

and 8 for $\rho r \geq \frac{15}{2}$. In particular, $\delta_N^{(1,1)}$, $\delta_N^{(3,1)}$, $\delta_N^{(7/2,3/2)}$, $\delta_N^{(15/2,1)}$ and $\delta_N^{(15/4,2)}$ ought to be approximately 1.33, 2.30, 4.29, 8.00 and 8.00, respectively. These values of $\delta_N^{(\rho,r)}$ are given in the last row of Table 1.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\gamma_N^{(1,1)}$</th>
<th>$\gamma_N^{(3,1)}$</th>
<th>$\gamma_N^{(7/2,3/2)}$</th>
<th>$\gamma_N^{(15/2,1)}$</th>
<th>$\gamma_N^{(15/4,2)}$</th>
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<tbody>
<tr>
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<td>$\tilde{\delta}_N^{(1,1)}$</td>
<td>$\tilde{\delta}_N^{(3,1)}$</td>
<td>$\tilde{\delta}_N^{(7/2,3/2)}$</td>
<td>$\tilde{\delta}_N^{(15/2,1)}$</td>
<td>$\tilde{\delta}_N^{(15/4,2)}$</td>
</tr>
<tr>
<td>32</td>
<td>1.4 E - 2</td>
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<td>3.8 E - 8</td>
<td>1.7 E - 8</td>
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<tr>
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<td>1.2 E - 8</td>
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<td>4.5 E - 10</td>
<td>5.4 E - 12</td>
<td>7.9 E - 13</td>
<td>3.9 E - 12</td>
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<tr>
<td></td>
<td>1.74</td>
<td>5.28</td>
<td>9.19</td>
<td>12.13</td>
<td>12.13</td>
</tr>
</tbody>
</table>

Table 2. ($m = 3, \nu = \frac{2}{5}, \lambda = \frac{1}{2}, \eta_1 = \frac{5 - \sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5 + \sqrt{15}}{10}$)

In a similar way we obtain from (5.16) that $\tilde{\delta}_{N}^{(1,1)}$, $\tilde{\delta}_{N}^{(3,1)}$, $\tilde{\delta}_{N}^{(4,1)}$, and $\tilde{\delta}_{N}^{(4,2)}$ ought to be approximately 1.74, 5.28, 9.19, 12.13 and 12.13, respectively. These values of $\tilde{\delta}_{N}^{(\ast)}$ are given in the last row of Table 2.

As we can see from Tables 1 and 2, the numerical results are in good agreement with the theoretical estimates. In Table 2 only the decrease of $\gamma_{N}^{(1,1)}$ is faster than it is indicated by theoretical estimates: the predicted value for $\tilde{\delta}_{N}^{(1,1)}$ is equal to 1.74, but the current experiment gave for $\tilde{\delta}_{N}^{(1,1)}$ a stable value 2.30. This phenomenon notifies that the local order of convergence of proposed algorithms needs further theoretical and numerical study.

References


