On Dependence of Sets of Functions on the Mean Value of their Elements

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Abstract. The paper considers, for a given closed bounded set $M \subset \mathbb{R}^m$ and $K = (0,1)^n \subset \mathbb{R}^n$, the set $\mathcal{M} = \{ h \in L_2(K;\mathbb{R}^m) \mid h(x) \in M \text{ a.e. } x \in K \}$ and its subsets

$$\mathcal{M}(\hat{h}) = \left\{ h \in \mathcal{M} \mid \int_K h(x) dx = \hat{h} \right\}.$$

It is shown that, if a sequence $\{\hat{h}_k\} \subset coM$ converges to an element $h_k \in \mathcal{M}(\hat{h}_k)$ there is $h_k' \in \mathcal{M}(\hat{h}_0)$ such that $h_k' - h_k \to 0$ as $k \to \infty$. If, in addition, the set $M$ is finite or $M$ is the convex hull of a finite set of elements, then the multivalued mapping $\hat{h} \mapsto \mathcal{M}(\hat{h})$ is lower semicontinuous on $coM$.

Key words: multivalued mapping, subsets of functions with fixed mean value, continuous dependence.

1 Introduction

Most sets of admissible control functions in the theory of optimal control are given as sets of measurable functions with values from a given set: for a given reference domain $Q \subset \mathbb{R}^n$ and a given set $M \subset \mathbb{R}^m$ the set of admissible controls is defined as

$$\mathcal{M} = \left\{ h \text{ measurable in } Q \mid h(x) \in M \text{ a.e. } x \in Q \right\}.$$

Here $n$ and $m$ are arbitrary fixed positive integers.

Provided that $Q$ is a bounded domain and $M$ is a bounded and closed set, the set $\mathcal{M}$ can be split as $\mathcal{M} = \bigcup_{\hat{h} \in coM} \mathcal{M}(\hat{h})$, where

$$\mathcal{M}(\hat{h}) := \left\{ h \text{ measurable, } h(x) \in M \text{ a.e. } x \in Q, \quad \frac{1}{|Q|} \int_Q h(x) dx = \hat{h} \right\}.$$

Here by $coA$ we denote the convex hull of the set $A$ and by $|Q|$ we denote the Lebesgue measure of the set $Q \subset \mathbb{R}^n$. 


Such a representation of $\mathcal{M}$ is useful when weak limits of sequences of control functions are involved, especially in procedures of relaxation via convexification, see, for instance, Warga [4]. Analogous splitting is used in the homogenization theory defining the so-called $G_\theta$-closures, see, for instance, Milton [2]. The corresponding relaxation procedures often involve the evaluation of integrals (over the periodicity cell $K = (0,1)^n$) of the kind

$$I(\hat{h}) = \inf_{h \in \mathcal{M}(\hat{h})} \int_K f(x,h(x)) \, dx$$

and the investigation of continuity properties of the function $\hat{h} \to I(\hat{h})$. To do that, obviously, one needs to know certain properties of the dependence of sets $\mathcal{M}(\hat{h})$ on $\hat{h}$.

In Sections 2 and 3 we shall show the following results.

**Theorem 1.** Let $Q \subset \mathbb{R}^n$ be bounded Lipschitz domain and let the set $M \subset \mathbb{R}^m$ is bounded and closed. Then for every given sequences $\{\hat{h}_k\}$ and $\{h_k\}$ such that

(i) $\{\hat{h}_k\} \subset \text{co}M$ and $\hat{h}_k \to \hat{h}_0$ in $\mathbb{R}^m$ as $k \to \infty$;

(ii) $h_k \in \mathcal{M}(\hat{h}_k)$, $k = 1, \ldots$,

there exists a sequence $\{h_{0k}\} \subset \mathcal{M}(\hat{h}_0)$ such that

$$h_k - h_{0k} \to 0 \text{ strongly in } L_2(Q;\mathbb{R}^m) \text{ as } k \to \infty.$$ 

**Theorem 2.** Let $Q \subset \mathbb{R}^n$ be bounded Lipschitz domain and let the set $M \subset \mathbb{R}^m$ is finite or $M$ is the closed convex hull of a finite set of elements. Then for every fixed sequence $\{\hat{h}_k\} \subset \text{co}M$ that converges in $\mathbb{R}^m$ to an element $\hat{h}_0$ and for every given element $h_0 \in \mathcal{M}(\hat{h}_0)$ there exists a sequence $\{h_k\}$, $h_k \in \mathcal{M}(\hat{h}_k)$, $k = 1,2,\ldots$, such that

$$h_k - h_0 \to 0 \text{ strongly in } L_2(Q;\mathbb{R}^m) \text{ as } k \to \infty.$$ 

**Remark 1.** Obviously, from Theorem 2 it follows immediately that the multivalued mapping $\hat{h} \to \mathcal{M}(\hat{h})$ is lower semicontinuous on $\text{co}M$ (for the definition and properties of multivalued mappings we refer to Kuratowski [1]).

**Remark 2.** It is easy to see that under hypotheses of Theorem 1 the function $\hat{h} \to I(\hat{h})$ defined by (1.1) is lower semicontinuous provided that $f$ is Caratheodory function and that $f$ has a majorant $f_0 \in L_1(Q)$ (we recall that the set $M$ is bounded). If, in addition, the hypotheses of Theorem 2 are satisfied, then the function $\hat{h} \to I(\hat{h})$ is continuous on $\text{co}M$.

### 2 Proof of Theorem 1

In this Section, we give the proof of Theorem 1. Since all reasonings below do not depend on concrete properties of the reference domain $Q$, then, without losing a generality, all proofs are given for the standard case $Q = K := (0,1)^n$. Since the set $M$ is bounded and closed, then the convex hull $\text{co}M$ of $M$ is
closed too and all sets \( M(\hat{h}) \) with \( \hat{h} \in coM \) are nonempty closed sets. In what follows, we shall use the notion of the relative interior \( riA \) for convex sets \( A \) from Euclidean spaces, for instance \( ricoM \) stands for the relative interior of the convex hull of \( M \). For the definition of \( riA \) and other notations and properties for convex sets we refer to Rockafellar [3]. Let \( r_0 \) be dimension of \( coM \).

**Step 1.** Let \( \hat{h}_0 \in ricoM \). Then there exists \( d > 0 \) such that \( \hat{h} \in ricoM \) whenever \( \hat{h} \in coM \) and \( |\hat{h} - \hat{h}_0| \leq d \). Let us fix \( \varepsilon > 0 \), \( 0 < \varepsilon < d/4 \), and let \( |\hat{h} - \hat{h}_0| \leq \varepsilon \). Then the element

\[
\hat{h}_* = \hat{h} + \frac{d}{\varepsilon}(\hat{h}_0 - \hat{h}) \in ricoM.
\]

Let \( h \in M(\hat{h}) \), \( h_* \in M(\hat{h}_*) \) be arbitrary chosen elements. By virtue of Lyapunov’s theorem on the range of vectorial measures for every \( \lambda \in [0, 1] \) there exists a measurable set \( E_\lambda \subset K \) such that

\[
|E_\lambda| = \lambda, \quad \int_{E_\lambda} h(y) \, dy + \int_{K \setminus E_\lambda} h_*(y) \, dy = \lambda \hat{h} + (1 - \lambda)\hat{h}_*.
\]

For a special choice \( \lambda = \lambda_0 = 1 - \varepsilon/d \) we define \( h_0 \) as

\[
h_0(\cdot) = \chi_{E_{\lambda_0}}(\cdot)h(\cdot) + (1 - \chi_{E_{\lambda_0}}(\cdot))h_*(\cdot),
\]

where \( \chi_E \) denotes the characteristic function of the set \( E \). By construction, \( h_0 \in M(\hat{h}_0) \) and

\[
\int_K (h(y) - h_0(y))^2 \, dy = \int_{K \setminus E_{\lambda_0}} (h(y) - h_0(y))^2 \, dy \leq 4c(M)\varepsilon/d,
\]

where \( c(M) \) depends only on \( M \). Thus, the assertion of Theorem 1 holds whenever \( \hat{h}_0 \in ricoM \).

**Step 2.** Let \( \hat{h}_0 \) does not belong to \( ricoM \). Because \( ricoM \) is not empty (provided that \( M \) consists of more than one element), then there exist a vector \( a \in \mathbb{R}^m \) and a constant \( c \) such that

\[
|a| = 1, \quad \langle a, \hat{h}_0 \rangle = c < \langle a, \hat{h} \rangle \quad \text{for all } \hat{h} \in ricoM.
\]

Without loosing generality, we can assume that \( c = 0 \), otherwise we can use the transform \( \hat{h} \mapsto \hat{h} - h_0 \).

Let \( M_1 = \{ h \in M \mid \langle a, h \rangle = 0 \} \). Because the sets \( M \) and \( M_1 \) are compact, then there exists a continuous function \( \gamma, \gamma(t) = 0 \) if \( t \leq 0 \), \( \gamma(t) > 0 \) if \( t > 0 \), such that

\[
\langle a, h - \hat{h}_0 \rangle \geq \gamma(\text{dist}\{h; M_1\}) \quad \text{for all } h \in M. \tag{2.1}
\]

Without loosing generality, we can assume that the function \( \gamma \) is convex, otherwise we can pass to the bipolar \( \gamma^{**} \), which has the desired properties. By construction, for nonnegative \( \tau \) there exists the inverse function \( \tau \mapsto \gamma^{-1}(\tau) \), \( \gamma^{-1}(\gamma(t)) = t \) for \( t \geq 0 \), which is continuous and strictly increasing on \( \{ \tau \in \mathbb{R}^m \mid \tau \geq 0 \} \).
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\( \{ h \in L_2(K; \mathbb{R}^r) \mid h(y) \in M_1 \text{ a.e. } y \in K \} \), such that

\[
\| h - h_* \|^2_{L_2(K; \mathbb{R}^r)} \leq c(m, M) \gamma^{-1}(\| \hat{h} - \hat{h}_0 \|)
\]

where \( c(m, M) \) depends only on \( m \) and \( M \). This way, for our situation with a fixed \( \hat{h}_0 \in \text{co}M_1 \), for every \( \hat{h} \in \text{co}M \) and arbitrary chosen \( h \in M(\hat{h}) \) there exists a corresponding \( h_* \in M_1 \) such that

\[
\| h - h_* \|^2_{L_2(K; \mathbb{R}^r)} \leq c(m, M) \gamma^{-1}(\| \hat{h} - \hat{h}_0 \|).
\]

By construction,

\[
\int_K h_*(y) \, dy = \hat{h}_* \in \text{co}M_1,
\]

\( M(\hat{h}_0) \subset M_1 \) and the dimension of \( \text{co}M_1 \) is less than \( r_0 \). From now on, we have to approximate the element \( h_* \in M(\hat{h}_*) \subset M_1 \) by elements from \( M(\hat{h}_0) \subset M_1 \), i.e. we have reduced the dimension \( r_0 \) of our problem to the problem with dimension less than or equal to \( r_0 - 1 \).

Step 3. To conclude our reasoning by induction over the dimension \( r_0 \) we have to prove our assertion for the case \( r_0 = 1 \). If \( \hat{h}_0 \in \text{rico}M \), then we apply reasoning from Step 1. If \( \hat{h}_0 \) does not belong to \( \text{rico}M \), then the set \( M_1 \) from the Step 2 consists of only one element \( \hat{h}_0 \) and the set \( M_1 \) consists of one constant function \( h_0(y) = \hat{h}_0 \text{ a.e. } y \in K \). For this case we can apply the same reasoning as in Step 2, what gives the statement of Theorem for \( r_0 = 1 \).

3 Proof of Theorem 2

In this Section, we give the proof of Theorem 2. Let \( M = \{ \overline{h}_1, \ldots, \overline{h}_N \} \subset \mathbb{R}^m \). Let \( H \) be \( m \times N \) matrix with columns \( \overline{h}_1, \ldots, \overline{h}_N \) respectively and let

\[
\Lambda := \left\{ \overline{\lambda} \in \mathbb{R}^N \mid \overline{\lambda} = (\lambda_1, \ldots, \lambda_N), \lambda_j \geq 0, j = 1, \ldots, N; \lambda_1 + \cdots + \lambda_N = 1 \right\}.
\]

To a given vector-function \( h \in M \) (it has only \( N \) admissible values from \( M \)) we can appoint an element \( \overline{\lambda} \) whose components \( \lambda_j \) represent the volume fractions
in $K$ of the sets where the vector-function $h$ has the value $\mathbf{1}_j$, $j = 1, \ldots, N$, respectively. Let $E := \{ \mathbf{r} \in \mathbb{R}^N | H\mathbf{r} = 0 \}$.

In these notations the statement of Theorem 2 is a straight consequence of:

$$
\begin{align*}
\text{if } \lambda_0 \in A, \{ \overline{\mathbf{v}}_k \} &\subset \mathbb{R}^N, \overline{\mathbf{v}}_k \to 0 \text{ as } k \to \infty \\
\text{and } \{ \lambda_0 + \overline{\mathbf{v}}_k + E \} &\cap A \neq \emptyset, k = 1, 2, \ldots, \\
\text{then there exists a sequence } \{ \lambda_k \} \text{ such that } \\
\lambda_k &\to \lambda_0 \text{ as } k \to \infty, \\
\lambda_k &\in \{ \lambda_0 + \overline{\mathbf{v}}_k + E \} \cap A, k = 1, 2, \ldots. 
\end{align*}
$$

(3.1)

Indeed, first of all we have to take care only about volume fractions of sets where the functions under consideration take the corresponding values $\mathbf{1}_j, \ldots, \mathbf{1}_N$ (we always can prearrange the corresponding sets preserving their measures).

Further, for $h \in \mathcal{M}$ with corresponding volume fractions $(\lambda_1, \ldots, \lambda_N) = \mathbf{\lambda}$ we have that $h \in \mathcal{M}(H\mathbf{\lambda})$, and every $h \in \text{co}M$ has the representation $\hat{h} = H(\lambda + E)$ with some $\lambda \in A$. The convergence $\overline{\mathbf{v}}_k \to 0$ as $k \to \infty$ in (3.1) implies the corresponding convergence $\lambda_k \to \lambda_0$ in Theorem 2, and the convergence $\lambda_k \to \lambda_0$ implies the corresponding convergence $h_k \to h_0$ in Theorem 2.

Let us denote $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^N$ and let us represent $E$ as the direct sum $E = E_0 \oplus E_1$ where

$$
E_0 = \{ \mathbf{r} \in E | \mathbf{r} \cdot \mathbf{1} = 0 \}.
$$

Here the subspace $E_1$ can be equal to $\{ 0 \}$ if $\mathbf{1}$ is orthogonal to $E$. From assumptions on $\overline{\mathbf{v}}_k$ we have the existence of $\overline{\mathbf{v}}_{0k} \in E_0$ and $\overline{\mathbf{v}}_{1k} \in E_1$ such that

$$
\lambda_0 + \overline{\mathbf{v}}_k = \overline{\mathbf{v}}_{0k} + \overline{\mathbf{v}}_{1k} \in A, \\
\overline{\mathbf{v}}_k + \overline{\mathbf{v}}_{1k} = 0, \overline{\mathbf{v}}_{1k} \to 0 \text{ as } k \to \infty.
$$

So, if necessary, using the transform $\overline{\mathbf{v}}_k \to \overline{\mathbf{v}}_k + \overline{\mathbf{v}}_{1k}$ and replacing $E$ by $E_0$, without losing generality, we can assume that

(i) the vector $\mathbf{1}$ is orthogonal to $E$;

(ii) $\langle \overline{\mathbf{v}}_k, \mathbf{1} \rangle = 0$, $k = 1, 2, \ldots$.

That means (since $\langle \lambda_0, \mathbf{1} \rangle = 1$) our further reasoning concerns only the hyper-plane $\{ \mathbf{r} \in \mathbb{R}^N | \mathbf{r} \cdot \mathbf{1} = 1 \}$. There are two possibilities:

(a) $\langle \lambda_0 + E \rangle \cap riA \neq \emptyset$;

(b) $\lambda_0$ belongs to a façade $A_s$ of $A$ with the dimension $s$, $0 \leq s \leq N - 2$, and $\lambda_0 + E$ can be separated from $riA$.

For the case (a) there exists a $\lambda_* \in (\lambda_0 + E) \cap riA$ and the elements

$$
\lambda_k = \lambda_0 + \overline{\mathbf{v}}_k + \tau_k(\mathbf{1}_* - \lambda_0), \quad k = 1, 2, \ldots,
$$

with appropriate $\tau_k > 0$, $k = 1, 2, \ldots$, solve the problem for $k \geq k_0$ with some $k_0$.

For the case (b), without losing generality, we can assume that \( A_s \) is the façade with the minimal dimension \( s \) compared to all façades, which contain \( \hat{\lambda}_0 \). Hence, after relabeling indexes we obtain

\[
A_s = \{ \mathbf{X} \in A \mid \mathbf{X} = (\lambda_1, \ldots, \lambda_N), \lambda_{s+2} = \cdots = \lambda_N = 0 \},
\]

\[
\hat{\lambda}_0 = (\lambda_1^0, \ldots, \lambda_N^0), \quad 0 < \lambda_1^0, \ldots, 0 < \lambda_{s+2}^0, \quad \lambda_{s+2}^0 = \cdots = \lambda_N^0 = 0.
\]

If

\[
\hat{\lambda}_0 + z_k + \overline{z}_k \in A \quad \& \quad \overline{z}_k \in E, \quad k = 1, 2, \ldots,
\]

then from

\[
\overline{a}_k \to 0 \quad \text{as} \quad k \to \infty, \quad \text{and} \quad \langle \overline{z}_k, \mathbf{1} \rangle = 0, k = 1, 2, \ldots,
\]

it follows immediately that the sequence \( \{\overline{z}_k\} \) is bounded.

Let us assume that the sequence \( \{\overline{z}_k\} \) converges to an element \( \overline{z}_0 \). If \( \overline{z}_0 = 0 \), then the sequence \( \overline{\lambda}_k := \hat{\lambda}_0 + \overline{a}_k + \overline{z}_k, \quad k = 1, 2, \ldots \)

solves the problem.

If \( \overline{z}_0 \neq 0 \), then those entries of \( \overline{z}_0 = (z_{01}, \ldots, z_{0N}) \), which are different from zero, are positive for \( j \geq s + 2 \) and negative for those indexes \( j' \), for which \( \lambda_j^{0'} = 1 \) (if any). Therefore, there exists \( d_0 > 0 \) such that \( \hat{\lambda}_0 + \tau \overline{z}_0 \in A \) provided \( 0 \leq \tau \leq d_0 \).

Since \( \overline{z}_k - \overline{z}_0 \to 0 \) as \( k \to \infty \), then the elements

\[
\overline{\lambda}_k := \hat{\lambda}_0 + \tau_k \overline{z}_0 + (\overline{z}_k - \overline{z}_0) + \overline{a}_k
\]

for \( k \geq k_0 \) and with appropriate \( \tau_k \), \( \tau_k \to 0 \) as \( k \to \infty \), belong to \( A \). Indeed, since \( \langle \overline{z}_0, \mathbf{1} \rangle = 0, \langle \overline{a}_k, \mathbf{1} \rangle = 0, \langle \overline{z}_k, \mathbf{1} \rangle = 0, \quad k = 1, 2, \ldots \), we have to check only inequalities

\[
\lambda_{kj} \geq 0, \quad j = 1, \ldots, N; \quad k = k_0, k_0 + 1, \ldots
\]

(obviously, \( \overline{\lambda}_k \to \hat{\lambda}_0 \) as \( k \to \infty \)). For those indexes \( \{j'\} \), for which entries of \( \overline{z}_0 \) are equal to zero,

\[
\lambda_j^{0'} + a_{kj'} + (z_{kj'} - z_{0j'}) \geq 0, \quad k = 1, 2, \ldots,
\]

(by the initial assumptions on the sequence \( \{\overline{z}_k\} \)), but for the rest of indexes \( \{j''\} \) either

\[
1 > \lambda_j^{0''} > 0
\]

or

\[
\lambda_j^{0''} = 1 \quad \& \quad z_{0j''} < 0
\]

what is sufficient for the existence of \( \tau_k \) with desired properties.

The general case of an arbitrary sequence \( \{\overline{z}_k\} \) is treated by standard reasoning by contradiction, i.e., we assume the contrary that there exist \( d > 0 \) and a sequence of indexes \( \{k'\} \) such that the distance from \( \hat{\lambda}_0 \) to \( \{\hat{\lambda}_0 + \overline{a}_{k'} + E\} \cap A \)....
is completed. After that we take an arbitrary subsequence of \( \{\tilde{h}_k\} \), for which the corresponding sequence \( \{T_k\} \) converges. The proof of the first part of Theorem 2 is completed.

Now, let \( M \) be closed convex hull of a finite number of elements \( \{h_1, \ldots, h_N\} \) and let

\[
S := \{\sigma \in L_2(K;\mathbb{R}^N) \mid \sigma = (\sigma_1, \ldots, \sigma_N), 0 \leq \sigma_j(x) \leq 1, j = 1, \ldots, N; \sum_{j=1}^N \sigma_j(x) = 1 \ a.e.\ x \in K\}.
\]

Since the function

\[
(\sigma, x) \rightarrow (h(x) - \sum_{j=1}^N \sigma_j h_j)^2
\]

is a normal integrand on \( \Lambda \times K \) (for every fixed \( h \in \mathcal{M} \)), then every \( h \in \mathcal{M} \) has the representation

\[
h(x) = \sum_{j=1}^N \sigma_j(x)h_j \ a.e. \ x \in K
\]

with some \( \sigma \in S \). In turn, a subset of piecewise constant elements is dense in \( S \) and sets \( \mathcal{M}(\hat{h}) \) have the same property.

This way, by using Cantor’s diagonal process, we have that it is sufficient to show the existence of the approximating sequence \( \{h_k\} \) for the case of a piecewise element \( h_0 \in \mathcal{M}(\hat{h}_0) \). Let \( Q_i \subset K, i = 1, \ldots, s \), are the sets where the function \( h_0 \) is constant and takes values \( g_1, \ldots, g_s \) from \( M \) respectively. Now, we define the set \( \hat{M} := \{h_1, \ldots, h_N, g_1, \ldots, g_s\} \) and sets

\[
\hat{\mathcal{M}}(\hat{h}) := \{h \text{ measurable in } K \mid h(x) \in \hat{M} \ a.e. \ x \in K, \int_K h(x)dx = \hat{h}\}.
\]

By construction, \( \text{co}\hat{M} = M \) and \( \hat{\mathcal{M}}(\hat{h}) \subset \mathcal{M}(\hat{h}) \forall \hat{h} \in M \).

If \( \{h_k\} \subset M \) and \( h_k \to \hat{h}_0 \) as \( k \to \infty \) then also \( \{\hat{h}_k\} \subset \text{co}\hat{M} \), \( \hat{h}_0 \in \text{co}\hat{M} \) and \( h_0 \in \mathcal{M}(\hat{h}_0) \). This way, the existence of the desired approximating sequence \( \{h_k\} \) follows immediately from the proof of the first part of Theorem 2. The proof of Theorem 2 is completed.

We conclude with a simple example, which shows that the statement of Theorem 2 is not, in general, true under hypotheses of Theorem 1. Let

\[
M = \{(-1, 0, 0), (1, 0, 0), (0, t, t^2), 0 \leq t \leq 1\} \subset \mathbb{R}^3.
\]

By construction,

\[
\mathcal{M}((0, t, t^2)) = \{(h_1, h_2, h_3) \in L_2(K;\mathbb{R}^3) \mid h_1(x) = 0, h_2(x) = t, h_3(x) = t^2; x \in K\} \text{ for } 0 < t < 1,
\]

\[
\mathcal{M}((0, 0, 0)) = \{(h_1, h_2, h_3) \in L_2(K;\mathbb{R}^3) \mid h_1(x) = -1 \text{ or } 0 \text{ or } 1, \int_K h_1(x)dx = 0; h_2(x) = 0, h_3(x) = 0; x \in K\},
\]

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and the statement of Theorem 2 does not hold.

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