STRONG $\mu$-FASTER CONVERGENCE AND STRONG $\mu$-ACCELERATION OF CONVERGENCE BY REGULAR MATRICES

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Abstract. The present paper continues the study of acceleration of convergence started in the paper [A. Aasma, Proc. Estonian Acad. Sci. Phys. Math., 2006, 55, 4, 195-209]. The new, non-classical convergence acceleration concept, called strong $\mu$-acceleration of convergence ($\mu$ is a positive monotonically increasing sequence), is introduced. It is shown that this concept allows to compare the speeds of convergence for a larger set of sequences than the classical convergence acceleration concept. Regular matrix methods are used to accelerate the convergence of sequences.

Key words: Convergence acceleration, matrix methods, speed of convergence.

1 Introduction

The present paper continues the study of acceleration of convergence of real or complex sequences started in [1]. Therefore all the notions not defined in this paper can be found in [1]. Throughout the paper we assume that indices and summation indices are integers, changing from 0 to $\infty$, if not specified otherwise.

Classically the convergence acceleration is defined as follows (cf. [5, 6]).

Definition 1. Let $x = (x_k)$ and $y = (y_k)$ be convergent sequences with limits $\varsigma$ and $\xi$, respectively. If

$$\lim_n \frac{|y_n - \xi|}{|x_n - \varsigma|} = 0,$$

then it is said that $y$ converges faster than $x$.

Definition 2. The sequence transformation $T : x \to y$ is said to accelerate the convergence of the sequence $x$ if $y$ converges faster than $x$. 
Some methods, alternative to the classical concept of estimation and comparison of speeds of convergence of sequences, are used in [3, 4]. In [1] another alternative method is proposed, where the concept, called $\mu$-faster convergence ($\mu$ is a positive monotonically increasing sequence) is introduced. It is shown that this concept allows the comparison of speeds of convergence for a larger set of sequences than the classical concept and this comparison is more precise.

Let $A = (a_{nk})$ be a matrix with real or complex entries. A sequence $x = (x_k)$ is said to be $A$-summable if the sequence $Ax = (A_n x)$ is convergent, where

$$A_n x = \sum_k a_{nk} x_k.$$  

We denote the set of all $A$-summable sequences by $c_A$. Thus, a matrix $A$ determines the summability method on $c_A$, which we also denote by $A$.

A method $A$ is said to be regular if for each $x = (x_n) \in c$, where $c$ is the set of all convergent sequences, the equality $\lim_n A_n x = \lim_n x_n$ holds. The convergence acceleration and $\mu$-acceleration of convergence by regular matrix methods were studied correspondingly in [3, 4, 5, 6, 7] and [1].

In the present paper the concept of strong $\mu$-faster convergence is defined and compared with the usual faster convergence concept, determined by Definitions 1 and 2. It is shown here that the new concept allows the comparison of speeds of convergence for a larger set of sequences than the classical concept and this comparison is more precise. It is also proved that if for a sequence $x = (x_n)$ with the limit $\zeta$ the sequence of absolute differences $(|x_n - \zeta|)$ is monotonically decreasing, then the strong $\mu$-faster convergence of a sequence $y$ with respect to $x$ coincides with the usual faster convergence of $y$ with respect to $x$. Also the concept of strong $\mu$-acceleration of convergence by a regular matrix method is defined and its properties are studied.

2 Main Results

Let $\varphi$ be a set of sequences such that

$$\varphi = \{ x = (x_k) \mid x_k = \text{const, if } k > k_0 \}$$  

for some $k_0 \geq 0$. For every sequence $x \in c \setminus \varphi$ we denote

$$\mu_x = \{ \mu = (\mu_n) \mid 0 < \mu_n \not\to \infty, \ l_n = \mu_n |x_n - \lim_n x_n| = O(1), \ l_n \not\to o(1) \}.$$  

Let us remind some notions from [1]. The sequence $\mu$ is called a speed of convergence of $x$. A sequence $\mu^* = (\mu^*_n) \in \mu_x$ is called the limit speed of convergence of $x$ if for all $\mu = (\mu_n) \in \mu_x$ the relation $\mu_n/\mu^*_n = O(1)$ holds. The limit speed of convergence $\mu^* = (\mu^*_n)$ of a sequence $y$ is said to be higher than the limit speed of convergence $\lambda^* = (\lambda^*_n)$ of a sequence $x$ if the ratio $\lambda^*_n/\mu^*_n$ is upper-bounded, but not lower-bounded. It is said that a sequence $y$ converges $\mu$-faster than $x$ if the limit speed of convergence of $y$ is higher than the limit speed of convergence of $x$ or $y \in \varphi$ and $x$ does not belong to $\varphi$.

Now we introduce the concept of strong $\mu$-faster convergence.
DEFINITION 3. Let \( \lambda^* = (\lambda^*_n) \) and \( \mu^* = (\mu^*_n) \) be correspondingly the limit speeds of convergence of convergent sequences \( x \) and \( y \). We say that \( y \) converges strongly \( \mu \)-faster than \( x \), if \( \lambda^*_n/\mu^*_n \longrightarrow 0 \) or \( y \in \phi \) and \( x \) does not belong to \( \phi \).

Remark 1. It is easy to see that if \( y \) converges strongly \( \mu \)-faster than \( x \), then \( y \) converges also \( \mu \)-faster than \( x \), but not vice versa. If \( y = (y_n) \) converges strongly \( \mu \)-faster than \( x = (x_n) \), then \( \lambda^*_n|y_n - \xi| = o(1) \), where \( \lambda^* = (\lambda^*_n) \) is the limit speed of \( x \) and \( \xi \) is the limit of \( y \). But for the case, if \( y \) converges only \( \mu \)-faster, but not strongly \( \mu \)-faster than \( x \), there exists a subsequence \( (y_{n_k}) \) of \( y \) so that \( \lambda^*_n|y_{n_k} - \xi| \neq o(1) \).

It was proved in [1] that if a sequence \( y = (y_n) \in c \) converges faster than \( x = (x_n) \in c \setminus \varphi \), then \( y \) converges also \( \mu \)-faster than \( x \). We prove that the similar property holds for the concept of strong \( \mu \)-faster convergence.

Theorem 1. If a sequence \( y = (y_n) \in c \) converges faster than \( x = (x_n) \in c \setminus \varphi \), then \( y \) converges also strongly \( \mu \)-faster than \( x \).

Proof. For \( y \in \varphi \) the assertion of Theorem 1 is clearly true. Thus, suppose that \( y \in c \setminus \varphi \) converges faster than \( x \in c \setminus \varphi \), i.e. relation (1.1) holds, and show that then \( y \) converges also strongly \( \mu \)-faster than \( x \). By Corollary 2.1 of [1] there exists the limit speed of convergence \( \lambda^* = (\lambda^*_n) \in \lambda_x \) of \( x \). Using relation (1.1) we have now

\[
\lim_n \frac{\lambda^*_n|y_n - \xi|}{\lambda^*_n|x_n - \varsigma|} = 0.
\]

Consequently, by Proposition 2.1 from [1] there exists \( \vartheta = (\vartheta_n), 0 < \vartheta_n \not\longrightarrow \infty \), such that

\[
\vartheta_n \frac{\lambda^*_n|y_n - \xi|}{\lambda^*_n|x_n - \varsigma|} = O(1).
\]

Denoting \( \vartheta_n\lambda^*_n = \mu_n \), we get from the last relation that \( \mu_n|y_n - \xi| = O(1) \) with \( 0 < \mu_n \not\longrightarrow \infty \) and \( \mu_n/\lambda^*_n \longrightarrow \infty \). Consequently for the limit speed of convergence \( \mu^* = (\mu^*_n) \) of \( y \) we have \( \mu^*_n/\lambda^*_n \longrightarrow \infty \). Thus \( y \) converges strongly \( \mu \)-faster than \( x \) by Definition 3. ■

The opposite assertion to Theorem 1, however, is not valid.

Example 1. Let \( x = (x_n) \in c \setminus \varphi \) be given by the relations

\[
x_n = \begin{cases} 
\frac{1}{(n + 1)2^n} & \text{if } n = 3k, \\
(n + 1)^38^n x_n = o(1) & \text{if } n = 3k + 1, \\
2^n(n + 1)^2x_n \neq O(1), & \text{if } n = 3k + 2,
\end{cases}
\]

where \( k = 0, 1, \ldots \). It was proved in [1] that applying Aitken’s process to the subsequence \( (x_{3k}) \) of \( x \) we get the sequence \( y = (y_n) \), where

\[
y_n = \frac{9}{8^n (1323n^3 + 6993n^2 + 12024n + 6736)}.
\]

A. Aasma

It is easy to see now that $y$ converges not faster than $x$ and $x$ converges not faster than $y$, but $y$ converges strongly $\mu$-faster than $x$. Indeed, we can determine the limit speeds of convergence of $x$ and $y$ respectively by $\lambda^* = (\lambda_n^*)$ and $\mu^* = (\mu_n^*)$, where

$$
\lambda_n^* = 2^n(n + 1), \quad \mu_n^* = 2^{3n}(n + 1)^3.
$$

As $\mu_n^*/\lambda_n^* \rightarrow \infty$, then $y$ converges strongly $\mu$-faster than $x$ by Definition 3.

Suppose now that $x = (x_n) \in c \setminus \varphi$ with the limit $\varsigma$ be a sequence for which the sequence of absolute differences $(|x_n - \varsigma|)$ is monotonically decreasing. We show that in this case the strong $\mu$-faster convergence coincides with the classical faster convergence.

**Theorem 2**. Let $x = (x_n) \in c \setminus \varphi$ be a sequence (with the limit $\varsigma$), for which the sequence of absolute differences $(|x_n - \varsigma|)$ is monotonically decreasing. If a sequence $y = (y_n)$ (with limit $\xi$) converges strongly $\mu$-faster than $x$, then $y$ converges also faster than $x$.

**Proof** It is not difficult to see that the limit speed $\lambda^* = (\lambda_n^*)$ of a sequence $x$ can be defined by the equality $\lambda_n^* = 1/|x_n - \varsigma|$. If $\mu^* = (\mu_n^*)$ is the limit speed of $y$, then we get

$$
\frac{\mu_n^* |y_n - \xi|}{\lambda_n^* |x_n - \varsigma|} = \mu_n^* |y_n - \xi| = O(1).
$$

Last relation implies equality (1.1), since $\mu_n^*/\lambda_n^* \rightarrow \infty$. ■

It is said (see [1]) that a regular method $A$ $\mu$-accelerates the convergence of a sequence $x \in c$ if the sequence $Ax$ converges $\mu$-faster than $x$.

**Definition 4**. We say that a matrix method $A$ strongly $\mu$-accelerates the convergence of a sequence $x \in c$ if the sequence $Ax$ converges strongly $\mu$-faster than $x$.

**Theorem 3**. For every $x \in c \setminus \varphi$ there exists a regular matrix $A$, which strongly $\mu$-accelerates the convergence of $x$.

**Proof** By Corollary 2.1 of [1] every $x \in c \setminus \varphi$ has the limit speed $\lambda^* = (\lambda_n^*)$. We show that there exists a regular matrix $A$ so that the limit speed of the sequence $(Ax_n)$ is higher than $\lambda^*$. As every $x = (x_n) \in c$ (with limit $\varsigma$) can be presented in the form

$$
x = x^0 + \varsigma e, \quad \text{where} \quad x^0 = (x^0_n) \in c_0 \quad \text{and} \quad e = (1, 1, ...),
$$

where $c_0$ is the set of sequences, converging to zero, then we get

$$
\lambda_n^* |x_n - \varsigma| = \lambda_n^* |x^0_n| = O(1) \quad \text{or} \quad |x^0_n| = O\left(\frac{1}{\lambda_n^*}\right) \quad \text{and} \quad \lambda_n^* |x^0_n| \neq o(1).
$$
As the limit speed $\lambda^*$ is a monotonically increasing unbounded sequence, then there exists a subsequence $(\lambda_{k_n}^*)$ of $\lambda^*$ such that $\lambda_{k_n}^* / \lambda_n^* \to \infty$. We define a matrix $A = (a_{nk})$ by the equalities

$$a_{nk} = \begin{cases} 1 & k = k_n, \\ 0 & k \neq k_n. \end{cases}$$

With the help of Theorem 2.3.7 from [2] it is not difficult to check that the matrix $A$ is regular. Now we have

$$\left| A_n x^0 \right| = \left| \sum_k a_{nk} x_k^0 \right| = \left| x_{k_n}^0 \right| = \mathcal{O}\left( \frac{1}{\lambda_{k_n}^*} \right)$$
or, equivalently,

$$\lambda_{k_n}^* \left| A_n x^0 \right| = \mathcal{O}(1).$$

Denoting $\mu = (\mu_n) = (\lambda_{k_n}^*)$, we get

$$\mu_n \left| A_n x^0 \right| = \mathcal{O}(1), \text{ where } \mu_n / \lambda_{k_n}^* \to \infty.$$

Therefore $A$ strongly $\mu$-accelerates the convergence of $x^0$. As $A_n e = 1$, then with the help of (2.1) we conclude

$$\mu_n \left| A_n x - \varsigma \right| = \mu_n \left| A_n x^0 + \varsigma A_n e - \varsigma \right| = \mu_n \left| A_n x^0 \right|.$$ 

Consequently $A$ strongly $\mu$-accelerates also the convergence of $x$. □

We note that the assertion of Theorem 3 does not hold for the concept of classical faster convergence. Indeed, it is not possible to accelerate the convergence of $x = (x_n) \in c \setminus \varphi$ by any regular matrix method if, for example, $x$ is defined by the relation

$$x_n = \begin{cases} 1 & n = 2k, \\ n + 1 & n = 2k + 1. \end{cases}$$

It follows from the proof of Theorem 3.2 of [1] that for every triangular regular matrix $A$ there exists a convergent sequence $x$, which converges $\mu$-faster than its $A$-transform $Ax$. For strong $\mu$-acceleration of convergence we can extract from the proof of Theorem 3.2 of [1] the following result.

**Proposition 1.** If a triangular regular matrix $A$ has a column with infinite number of non-zero elements, then there exists a sequence $x$, converging strongly $\mu$-faster than its $A$-transform $Ax$.

As we see from Proposition 1, for some triangular regular methods $A$ it is possible to choose a sequence $x$, converging strongly $\mu$-faster than its $A$-transform $Ax$, but it is not so for all triangular regular matrices.

Example 2. Let $A = (a_{nk})$ be defined by the relation
\[
a_{nk} = \begin{cases} 
\delta_{nk} & n = 2j, \\
\frac{1}{2} & n = 2j + 1, \ k = n - 1, n, \\
0 & k < n - 1,
\end{cases}
\]
where $j = 0, 1, \ldots$. Then for every convergent sequence $x = (x_k)$ we get
\[
A_n x = \begin{cases} 
x_n & n = 2j, \\
\frac{1}{2} (x_{n-1} + x_n) & n = 2j + 1,
\end{cases}
\]
where $j = 0, 1, \ldots$. Now a sequence $x$ can converge $\mu$-faster than its $A$-transform $Ax$ only in the case, if $x_{n-1}/x_n \neq \mathcal{O}(1)$. But never $x$ can converge strongly $\mu$-faster than its $A$-transform $Ax$.

References


