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ROBUST NUMERICAL METHOD FOR A SYSTEM OF SINGULARLY PERTURBED PARABOLIC REACTION–DIFFUSION EQUATIONS ON A RECTANGLE [1]

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Abstract. A Dirichlet problem is considered for a system of two singularly perturbed parabolic reaction–diffusion equations on a rectangle. The parabolic boundary layer appears in the solution of the problem as the perturbation parameter ε tends to zero. On the basis of the decomposition solution technique, estimates for the solution and derivatives are obtained. Using the condensing mesh technique and the classical finite difference approximations of the boundary value problem under consideration, a difference scheme is constructed that converges ε-uniformly at the rate $\mathcal{O}\left(N^{-2}\ln^2 N + N_0^{-1}\right)$, where $N = \min_s N_s$, $s = 1, 2$, $N_s + 1$ and $N_0 + 1$ are the numbers of mesh points on the axis x_s and on the axis t, respectively.

Key words: initial–boundary value problem, problem on a rectangle, perturbation parameter ε, system of parabolic equations, reaction–diffusion equations, finite difference approximation, parabolic boundary layer, decomposition solution technique, *a priori* estimates, ε-uniform convergence.

1 Introduction

In the present paper, finite difference approximations for a Dirichlet problem are considered for a system of two singularly perturbed parabolic reaction–diffusion equations on a rectangle. The highest-order derivatives in the differential equations are multiplied by the perturbation parameter ε^2; the parameter ε takes arbitrary values in the open-closed interval $(0, 1]$. For $\varepsilon = 0$, the system

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of the parabolic second-order equations degenerates into a system of ordinary differential equations. Equations in the system are coupled by terms that do not include derivatives. For small values of ε, a parabolic boundary layer appears in a neighbourhood of the boundary. Using the condensing mesh technique and the classical finite difference approximations of the boundary value problem, we construct a difference scheme that converges ε-uniformly. A description of such a method one can found, e.g., in [1, 2, 5, 7, 9].

To analyze conditions that ensure the required smoothness of the solution of the boundary value problem, and when deriving *a priori* estimates, and justifying the convergence of the special difference scheme, we apply a technique similar to that of [12] which considers a system of singularly perturbed elliptic equations on a rectangle with the vector perturbation parameter $\overline{\varepsilon}$. Here, unlike [12], the perturbation parameter ε is scalar.

2 Problem Formulation. The Aim of Research

2.1. In the domain $\overline{G}$:

$$\overline{G} = G \bigcup S, \quad G = D \times (0,\, T], \tag{2.1a}$$

where $\overline{D}$ is the rectangle[1]

$$\overline{D} = D \bigcup \Gamma, \quad D = D_{(2.1)} = \{x: \quad 0 < x_s < d_s, \quad s = 1, 2\}, \tag{2.1b}$$

we consider the Dirichlet problem for a system of singularly perturbed parabolic equations

$$L\,\mathbf{u}(x,\, t) \;=\; \mathbf{f}(x,\, t), \quad (x,\, t) \in G, \tag{2.2a}$$

$$\mathbf{u}(x,\, t) \;=\; \boldsymbol{\varphi}(x,\, t), \quad (x,\, t) \in S. \tag{2.2b}$$

Here $S = S^L \bigcup S_0$, where S^L and S_0 are the lateral and lower parts of the boundary S, $S^L = \Gamma \times (0,\, T]$, $S_0 = \overline{S}_0$;

$$L\,\mathbf{u}(x,\, t) = L(\varepsilon)\,\mathbf{u}(x,\, t) \equiv \left\{ \varepsilon^2\, L_2 - C(x,\, t) - P(x,\, t)\, \frac{\partial}{\partial t} \right\} \mathbf{u}(x,\, t),$$

$$L_2 = \begin{pmatrix} L_2^1 & 0 \\ 0 & L_2^1 \end{pmatrix}, \quad L_2^1 = \sum_{s=1,2} \frac{\partial^2}{\partial x_s^2},$$

$$C(x,\, t) = \begin{pmatrix} c^{11}(x,\, t) & c^{12}(x,\, t) \\ c^{21}(x,\, t) & c^{22}(x,\, t) \end{pmatrix}, \quad P(x,\, t) = \begin{pmatrix} p^1(x,\, t) & 0 \\ 0 & p^2(x,\, t) \end{pmatrix},$$

and $\mathbf{u}(x,\, t)$, $\mathbf{f}(x,\, t)$ and $\boldsymbol{\varphi}(x,\, t)$ are vector functions, for example,

$$\mathbf{u}(x,\, t) = (u^1(x,\, t),\, u^2(x,\, t))^T, \quad (x,\, t) \in \overline{G}.$$

[1] The notation $L_{(j.k)}$ ($\overline{G}_{(j.k)}, M_{(j.k)}$) means that these operators (domains, constants) are introduced in formula $(j.k)$.

We shall use both the vector form of the boundary value problem and the scalar form

$$\begin{aligned} L^i\,\mathbf{u}(x,\,t) &= f^i(x,\,t), \quad (x,\,t)\in G, \\ u^i(x,\,t) &= \varphi^i(x,\,t), \quad (x,\,t)\in S, \quad i=1,2, \end{aligned} \tag{2.2c}$$

where the operator $L^i = L^i_{(2.2c)}$ is defined by the relation

$$L^i\,\mathbf{u}(x,\,t) = \varepsilon^2\,L^1_2\,u^i(x,\,t) - \sum_{j=1,2} c^{ij}(x,\,t)\,u^j(x,\,t) - p^i(x,\,t)\,\frac{\partial}{\partial t}u^i(x,\,t).$$

The functions $c^{ij}(x,\,t)$, $p^i(x,\,t)$, $f^i(x,\,t)$, and $\varphi^i(x,\,t)$ are assumed to be sufficiently smooth on the set $\overline{G}$ and on the boundary S, respectively. Assume also that the following conditions are satisfied[2]

$$p_0 \le p^i(x,\,t) \le p^0, \quad (x,\,t)\in\overline{G}, \quad p_0>0; \tag{2.3a}$$

$$c^{ii}(x,\,t) \ge c_0, \quad mc^{ii}(x,\,t) \ge |c^{ij}(x,\,t)|, \quad (x,\,t)\in\overline{G}, \tag{2.3b}$$

$$i,j=1,2, \quad i\ne j, \quad c_0>0, \quad m=m_{(2.3)}<1.$$

The parameter ε takes arbitrary values in the open-closed interval $(0,1]$.

$$L^{(i)}_{(2.4)} \equiv \varepsilon^2\,L^1_2 - c^{ii}(x,\,t) - p^i(x,\,t)\,\frac{\partial}{\partial t}, \qquad (x,\,t)\in\overline{G}, \quad i=1,2, \tag{2.4}$$

that include the differential part of the operator $L^i_{(2.2c)}$ are monotone, moreover, the elements of the matric $C_{(2.2)}(x,t)$ have the strong diagonal dominance. Such properties of the operator $L_{(2.2)}$ allow us to estimate the solution using the data of the problem (2.2), (2.1).

$\mathbf{u}\in C^{2,1}(G)$ that is continuous on $\overline{G}$ and satisfies the differential equation (2.2a) on G and the boundary condition (2.2b) on S.

We assume that the solution of the problem is sufficiently smooth.

When ε tends to zero, a parabolic boundary layer appears in a neighbourhood of the set S^L.

2.2. Even in the case of the scalar reaction–diffusion equation, the classical difference schemes cannot produce ε-uniformly convergent solutions (see, e.g., [9]). Boundary value problems for systems of parabolic equations on a strip were considered, for example, in [10, 11] (for reaction–diffusion equations in [10], and for convection–diffusion equations in [11]); problems for systems in domains with piecewise smooth boundaries have not been studied.

Our aim for the boundary value problem (2.2), (2.1) is to construct a difference scheme that converges ε-uniformly.

3 Preliminary Considerations

3.1. Let us formulate the conditions imposed on the data of the problem (2.2), (2.1) that guarantee the required smoothness of the solution.

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[2] Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε and on the discretization parameters.

We denote by Γ_j, $\Gamma = \bigcup \Gamma_j$, $j = 1, 2, 3, 4$ the sides of the rectangle D; the sides Γ_s, Γ_{s+2} are orthogonal to the x_s-axis, for $s = 1, 2$; the sides Γ_1 and Γ_2 pass through the point $(0, 0)$, moreover, the Γ_j are closed sets; Γ^c is the set of corner points. Set

$$S_j = \Gamma_j \times (0, T], \quad j = 1, \dots, 4. \tag{3.1a}$$

We denote by S^c the set of "edges", i.e., S^c is formed by taking pairwise intersection of the faces in the parabolic boundary S (i.e., the lateral sides and lower basis of the set G):

$$S^c = S^{Lc} \bigcup S_0^c, \tag{3.1b}$$

where S^{Lc} and S_0^c are the set of the lateral and lower "edges" respectively:

$$\begin{aligned} &S^{Lc} = \textstyle\bigcup_{j=1}^{4} S_{j,j+1}^c, \quad S_{j,j+1}^c = S_j \bigcap S_{j+1}, \\ &S_0^c = \overline{S}^L \bigcap S_0, \quad S_{j+1} = S_1 \ \ \mathit{for} \ \ j = 4. \end{aligned} \tag{3.1c}$$

We give a **definition** of a compatibility condition on the set $S_{0(3.1)}^c$, following to [4]. Set $\boldsymbol{\varphi}_0(x) = \boldsymbol{\varphi}(x, t)$, $(x, t) \in S_0$. Let the function $\boldsymbol{\varphi}(x, t)$, $(x, t) \in S$, satisfy the condition $\boldsymbol{\varphi}(\cdot, 0) \in C^{l_0}(\overline{D})$ $(\boldsymbol{\varphi}_0 \in C^{l_0}(\overline{D}))$, and for the function $\boldsymbol{\varphi}(x, t)$, considered on $\overline{S}^L$, the derivatives $(\partial^{k_0}/\partial t^{k_0})\,\boldsymbol{\varphi}(x, t)$, where $(x, t) \in S_0^c$, are defined for $k_0 \le l/2$, $l_0 = l + \alpha$, $l \ge 0$ is the integer, $\alpha \in (0, 1)$. Using the function $\boldsymbol{\varphi}_0(x)$ prescribed on the set S_0 and the equation (2.2a), we find the derivative in t of the function $\mathbf{u}(x, t)$ on S_0. We denote it by $(\partial/\partial t)\,\boldsymbol{\varphi}_{0,t=0}(x)$. Furthermore, differentiating the equation (2.2a) in x_1, x_2 and t, we find the derivatives in t up to order $k_0 \le [l/2]$, where $[a]$ is the integer part of the number $a \ge 0$; we denote these derivatives by $(\partial^{k_0}/\partial t^{k_0})\,\boldsymbol{\varphi}_{0,t=0}(x)$, $x \in \overline{D}$. We say that the data of the boundary value problem satisfy a *compatibility condition on the set* S_0^c guaranteeing the continuity of the derivatives in t up to order K_0 of the function $\mathbf{u}(x,\, t)$, or, briefly, the problem data satisfy a *compatibility condition on* S_0^c *for the derivatives in* t *up to order* K_0 [4], if one has the condition

$$\frac{\partial^{k_0}}{\partial t^{k_0}}\,\boldsymbol{\varphi}(x, t) = \frac{\partial^{k_0}}{\partial t^{k_0}}\,\boldsymbol{\varphi}_{0,t=0}(x), \quad (x, t) \in S_0^c, \ \ 0 \le k_0 \le K_0. \tag{3.2}$$

Under the conditions given above, we have $K_0 \le [l/2]$, where $[l/2]$ is the integer.

In the case when the data of the problem (2.2), (2.1) satisfy the conditions

$$C_{(2.2)},\, P,\, \mathbf{f} \in H^{l^{(1)}+\alpha}(\overline{G}), \quad l^{(1)} \ge 0, \tag{3.3a}$$

$$\boldsymbol{\varphi} \in H^{l^{(2)}+\alpha}(\overline{S}_j), \quad \boldsymbol{\varphi} \in H^{l^{(2)}+\alpha}(S_0), \quad \boldsymbol{\varphi} \in C(S), \tag{3.3b}$$

$$j = 1, 2, 3, 4, \quad l^{(2)} \ge 2, \quad \alpha \in (0, 1),$$

then the solution of this problem satisfies the inclusion (see [3, 4]):

$$\mathbf{u} \in H^{l^{(3)}+\alpha_1}(G) \bigcap H^{\alpha_1}(\overline{G}), \quad \mathit{where} \ \ l^{(3)} = \min[l^{(1)} + 2, l^{(2)}], \quad \alpha_1 \in (0, 1).$$

Let

$$\begin{aligned}&\textit{the data of the problem (2.2), (2.1) on the set } S_0^c\\&\textit{satisfy compatibility conditions,}\\&\textit{for the derivatives in } t \textit{ up to order } [l^{(4)}/2], \quad l^{(4)} \le l^{(3)}.\end{aligned} \tag{3.3c}$$

Thereby, the derivatives in t satisfy the inclusion

$$\frac{\partial^{k_0}}{\partial t^{k_0}}\,\mathbf{u} \in H^{\alpha_1}(\overline{G}), \quad k_0 \le K_0, \quad \textit{where} \;\; K_0 \le [l^{(4)}/2], \quad l^{(4)} = l^{(4)}_{(3.3)}.$$

3.2. If the data of the problem (2.2), (2.1) satisfy the condition (3.3) with

$$l^{(1)} = l^{(2)} = l^{(4)} = l + 2, \quad l \ge 0, \tag{3.4a}$$

and also the conditions

$$\begin{aligned}&\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}\,P(x,\,t) = 0, \quad \frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}\,C(x,\,t) = 0, \quad (x,\,t) \in \overline{S}^{Lc},\\&\qquad 1 \le k \le 2([l/2]-1) \;\; \textit{for} \quad l \ge 4;\\&\textit{for } l \le 3 \;\; (l \ge 0), \textit{ restrictions to the derivatives}\\&\textit{of the matrix-functions } P(x,\,t), \;\; C(x,\,t), \;\; (x,\,t) \in \overline{S}^{Lc}, \textit{ are not imposed;}\end{aligned} \tag{3.4b}$$

$$\begin{aligned}&\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}\,\mathbf{f}(x,\,t) = \mathbf{0}, \quad k \le l,\\&\frac{\partial^{k_1}}{\partial x_1^{k_1}}\,\boldsymbol{\varphi}(x,\,t) = \frac{\partial^{k_2}}{\partial x_2^{k_2}}\,\boldsymbol{\varphi}(x,\,t) = \mathbf{0}, \quad k_1,\,k_2 \le l+2, \quad (x,\,t) \in \overline{S}^{Lc},\end{aligned} \tag{3.4c}$$

then one has $\mathbf{u}(\cdot,\,t) \in C^{l+2+\alpha}(\overline{D})$, $t \in [0,\,T]$, that implies the following inclusion [3, 4]:

$$\mathbf{u} \in H^{l+2+\alpha}(\overline{G}). \tag{3.5}$$

3.3. We shall assume that the following condition holds (we call it the condition (3.6)):

The data of the problem (2.2), (2.1) *satisfy the conditions* (3.3), (3.4) *that guarantee the smoothness of the solution of the boundary value problem on* $\overline{G}$, *i.e. the inclusion* (3.5). *When constructing a priori estimates for the regular and singular components of the solution in representations* (4.1), (4.8), (4.14) *(from Section* 4*), the following condition is assumed to be fulfilled in addition to the conditions* (3.3), (3.4):

$$\begin{aligned}&C_{(2.2)},\, P,\, \mathbf{f} \in H^{l_1+\alpha}(\overline{G}), \quad \boldsymbol{\varphi} \in H^{l_1+\alpha}(\overline{S}_j), \quad \boldsymbol{\varphi} \in C(S);\\&\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}\,\mathbf{f}(x,\,t) = \mathbf{0}, \quad (x,\,t) \in \overline{S}^{Lc}, \quad k \le l_1;\\&\frac{\partial^{k_1}}{\partial x_1^{k_1}}\,\boldsymbol{\varphi}(x,\,t) = \frac{\partial^{k_2}}{\partial x_2^{k_2}}\,\boldsymbol{\varphi}(x,\,t) = \mathbf{0}, \quad (x,\,t) \in \overline{S}^{Lc}, \quad k_1,\,k_2 \le l_1, \quad l_1 \ge l,\end{aligned} \tag{3.6}$$

that guarantee the smoothness of the regular and singular components of the solution.

The actual values of l and l_1 are specified where it is required. The fulfilment of other conditions in addition to (3.3), (3.4), (3.6) is not assumed.

4 A Priori Estimates of Solutions

4.1. Let us give some estimates that are obtained using main terms in the asymptotic expansion of the solution (see, e.g., [6, 9]).

Write the solution of the problem as the sum of the functions

$$\mathbf{u}(x,\,t) = \mathbf{U}(x,\,t) + \mathbf{V}(x,\,t), \quad (x,\,t) \in \overline{G}, \tag{4.1}$$

where $\mathbf{U}(x,\,t)$ and $\mathbf{V}(x,\,t)$ are the regular and singular parts of the solution. The function $\mathbf{U}(x,\,t)$, $(x,\,t) \in \overline{G}$, is the restriction to $\overline{G}$ of the function $\mathbf{U}^0(x,\,t)$, $(x,\,t) \in \overline{G}^{\,0}$, where the set $\overline{G}^{\,0}$, i.e., the extension of $\overline{G}$ beyond the $\overline{S}^L$, includes $\overline{G}$ along with its m_0-neighbourhood; $\overline{G}^{\,0} = \overline{D}^{\,0} \times [0,\,T]$. The function $\mathbf{U}^0(x,\,t)$ is the solution of the problem

$$\begin{aligned} L^0\,\mathbf{U}^0(x,\,t) &= \mathbf{f}^0(x,\,t), \quad (x,\,t) \in G^0, \\ \mathbf{U}^0(x,\,t) &= \boldsymbol{\varphi}^0(x,\,t), \quad (x,\,t) \in S^0. \end{aligned} \tag{4.2}$$

Here L^0 and $\mathbf{f}^0(x,\,t)$, $(x,\,t) \in \overline{G}^{\,0}$, are smooth continuations of the operator $L_{(2.2)}$ (that preserve the properties (2.3)) and of the function $\mathbf{f}(x,\,t)$; the function $\boldsymbol{\varphi}^0(x,\,t)$, $(x,\,t) \in S^0$, is chosen sufficiently smooth; $\boldsymbol{\varphi}^0(x,\,t) = \boldsymbol{\varphi}(x,\,t)$, $(x,\,t) \in S_0$. Assume that the functions $\mathbf{f}^0(x,\,t)$ and $\boldsymbol{\varphi}^0(x,\,t)$ are equal to zero outside an m_1-neighbourhood of the set $\overline{G}$, $m_1 < m_0$. The function $\mathbf{V}(x,\,t)$ is the solution of the problem

$$\begin{aligned} &L_{(2.2)}\,\mathbf{V}(x,\,t) = \mathbf{0}, \qquad (x,\,t) \in G, \\ &\mathbf{V}(x,\,t) = \boldsymbol{\varphi}(x,\,t) - \mathbf{U}(x,\,t) \equiv \boldsymbol{\varphi}_{\mathbf{V}}(x,\,t), \quad (x,\,t) \in S. \end{aligned} \tag{4.3}$$

4.2. Let us give estimates of the regular and singular components in the representation (4.1) of the solution of the boundary value problem.

4.2.1. Now we estimate the regular component of the solution. We represent the function $\mathbf{U}(x,\,t)$ as the sum of the functions

$$\mathbf{U}(x,t) = \sum_{k=0}^{n} \varepsilon^{2k}\mathbf{U}_k(x,t) + \mathbf{v}^n_{\mathbf{U}}(x,t) \equiv \mathbf{U}^n(x,t) + \mathbf{v}^n_{\mathbf{U}}(x,t),\ (x,t) \in \overline{G}, \tag{4.4}$$

corresponding to the representation of the function $\mathbf{U}^0(x,\,t)$, $(x,\,t) \in \overline{G}^{\,0}$, which is the solution of problem (4.2):

$$\mathbf{U}^0(x,\,t) = \sum_{k=0}^{n} \varepsilon^{2k}\mathbf{U}^0_k(x,\,t) + \mathbf{v}^{n0}_{\mathbf{U}}(x,\,t), \quad (x,\,t) \in \overline{G}^{\,0}.$$

The functions $\mathbf{U}_k^0(x,\,t)$, $(x,\,t) \in \overline{G}^{\,0}$, i.e., components in the decomposition of the regular part of the solution, are solutions of the problems

$$\begin{aligned} L_{(4.5)}\,\mathbf{U}_0^0(x,\,t) &= \mathbf{f}^0(x,\,t), \quad (x,\,t) \in \overline{G}^{\,0} \setminus S_0^0, \\ \mathbf{U}_0^0(x,\,t) &= \boldsymbol{\varphi}^0(x,\,t), \quad (x,\,t) \in S_0^0; \\ L_{(4.5)}\,\mathbf{U}_k^0(x,\,t) &= \varepsilon^{-2}\left\{L_{(4.5)} - L^0_{(4.2)}\right\}\mathbf{U}^0_{k-1}(x,\,t), \quad (x,\,t) \in \overline{G}^{\,0} \setminus S_0^0, \\ \mathbf{U}_k^0(x,\,t) &= \mathbf{0}, \quad (x,\,t) \in S_0^0, \quad k > 0, \end{aligned} \tag{4.5}$$

$$L_{(4.5)} = L^0_{(4.2)}|_{\varepsilon=0} = -C^0(x,\,t) - P(x,\,t)\,\frac{\partial}{\partial t}.$$

In the case of the condition (3.6), where

$$l \ge K - 2, \quad l_1 \ge K + 2\,n, \tag{4.6a}$$

$$n = [(K+1)/2]_{(3.3)} - 2, \quad K \ge 4, \tag{4.6b}$$

one has $\mathbf{U}^0 \in H^{K+\alpha}(\overline{G}^{\,0})$; for the function $\mathbf{U}(x,\,t)$, we obtain

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}\,\mathbf{U}(x,t)\right| \le M\left[1 + \varepsilon^{K-k-2}\right], \quad (x,t) \in \overline{G},\ k + 2k_0 \le K. \tag{4.7}$$

4.2.2. Let us consider the decomposition of the singular part.

In our constructions, we shall use a function $\mathbf{V}^d(x,\,t)$ (an approximation of $\mathbf{V}_{(4.1)}(x,\,t)$) as the singular term of the decomposition of the solution. We construct the function $\mathbf{V}^d(x,\,t)$ as the sum

$$\mathbf{V}^d(x,\,t) = \sum_{j=1}^{4}\left[\mathbf{V}_{(j)}(x,\,t) + \mathbf{V}_{(j,\,j+1)}(x,\,t)\right], \quad (x,\,t) \in \overline{G}, \tag{4.8}$$

where $\mathbf{V}_{(j,\,j+1)}(x,\,t) = \mathbf{V}_{(14)}(x,\,t)$ for $j = 4$. Here $\mathbf{V}_{(j)}(x,\,t)$ and $\mathbf{V}_{(j,\,j+1)}(x,\,t)$ are the functions describing the one-dimensional and the corner parabolic boundary layers in a neighbourhood of the sides S_j and the edges $S^c_{j,\,j+1} = S_j \bigcap S_{j+1}$, respectively, where $S_j \bigcap S_{j+1} = S^c_{1\,4}$ for $j = 4$. The functions $\mathbf{V}_{(j)}(x,\,t)$ and $\mathbf{V}_{(j,\,j+1)}(x,\,t)$, $(x,\,t) \in \overline{G}$, are the restrictions to $\overline{G}$ of the functions $\mathbf{V}^0_{(j)}(x,\,t)$, $(x,\,t) \in \overline{G}_{(j)}$ and $\mathbf{V}^0_{(j,\,j+1)}(x,\,t)$, $(x,\,t) \in \overline{G}_{(j,\,j+1)}$. Here

$$\overline{G}_{(j)} = \overline{D}_{(j)} \times [0,\,T], \quad \overline{G}_{(j,j+1)} = \overline{D}_{(j,j+1)} \times [0,\,T];$$

the set $\overline{D}_{(j)}$ (the set $\overline{D}_{(j,\,j+1)}$) is the part of the set $\overline{D}^{\,0}$, which along with the $\overline{D}$ belongs to the half-plane (the quarter–plane) whose boundary pass through the side Γ_j (the sides Γ_j and Γ_{j+1}).

The function $\mathbf{V}^0_{(j)}(x,\,t)$, $(x,\,t) \in \overline{G}_{(j)}$, is the solution of the problem

$$\begin{aligned} L^0\,\mathbf{V}^0_{(j)}(x,\,t) &= \mathbf{0}, \quad (x,\,t) \in G_{(j)}, \\ \mathbf{V}^0_{(j)}(x,\,t) &= \boldsymbol{\varphi}_{(j)}(x,\,t), \quad (x,\,t) \in S_{(j)}, \quad j = 1,2,3,4, \end{aligned} \tag{4.9a}$$

where $L^0 = L^0_{(4.2)}$; $\boldsymbol{\varphi}_{(j)}(x,\,t)$, $(x,\,t) \in S_{(j)}$, is a sufficiently smooth function that satisfies the condition

$$\boldsymbol{\varphi}_{(j)}(x,\,t) = \boldsymbol{\varphi}_{\mathbf{V}}(x,\,t), \quad (x,\,t) \in S_j \bigcup S_0. \tag{4.9b}$$

The function $\mathbf{V}^0_{(j)}(x,\,t)$ exponentially decreases when moving away from the set $S_{(j)j}$ (from the side of the boundary $S_{(j)}$ that includes S_j).

The function $\mathbf{V}^0_{(j,\,j+1)}(x,\,t)$, $(x,\,t) \in \overline{G}_{(j,\,j+1)}$, is the solution of the problem

$$\begin{aligned} L^0\,\mathbf{V}^0_{(j,\,j+1)}(x,\,t) &= \mathbf{0}, && (x,\,t) \in G_{(j,\,j+1)}, \\ \mathbf{V}^0_{(j,\,j+1)}(x,\,t) &= \boldsymbol{\varphi}_{(j,\,j+1)}(x,\,t), && (x,\,t) \in S_{(j,\,j+1)}, \quad j = 1,2,3,4; \end{aligned} \tag{4.10a}$$

$\boldsymbol{\varphi}_{(j,\,j+1)}(x,\,t)$, $(x,\,t) \in S_{(j,\,j+1)}$, is a sufficiently smooth function that satisfies the condition

$$\boldsymbol{\varphi}_{(j,\,j+1)}(x,\,t) = \boldsymbol{\varphi}_{\mathbf{V}}(x,\,t) - \boldsymbol{\varphi}_{(j)}(x,\,t) - \boldsymbol{\varphi}_{(j+1)}(x,\,t), \quad (x,\,t) \in S_j \bigcup S_{j+1}. \tag{4.10b}$$

The function $\mathbf{V}^0_{(j,\,j+1)}(x,\,t)$ exponentially decreases when moving away from the set $S_j \bigcap S_{j+1}$.

4.2.3. Let us estimate the singular components in the representation (4.8).

Having an estimate of the problem (4.9) in the extended domain $\overline{G}_{(j)}$ (the main term in the expansion of the form (4.4) of the function $\mathbf{V}^0_{(j)}(x,\,t)$, $(x,\,t) \in \overline{G}_{(j)}$, is a solution of the boundary value problem for a singularly perturbed vector one-dimensional parabolic equation), for the function $\mathbf{V}_{(j)}(x,\,t)$, $(x,\,t) \in \overline{G}$, in the case of the conditions (3.6), (4.6), we obtain the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1}\,\partial x_2^{k_2}\,\partial t^{k_0}}\,\mathbf{V}_{(j)}(x,\,t) \right| \le\; M\,\varepsilon^{-k_{(j)}}\,\exp\left(-m\,\varepsilon^{-1}\,r(x,\mathit{\Gamma}_j)\right), \quad (x,\,t) \in \overline{G}, \quad k + 2\,k_0 \le K, \quad j = 1,2,3,4. \tag{4.11}$$

Here $r(x,\mathit{\Gamma}_j)$ is the distance from the point x to the set $\mathit{\Gamma}_j$, $k_{(j)} = k_1$ for $j = 1,3$, while $k_{(j)} = k_2$ for $j = 2,4$, and m is an arbitrary constant from the interval $(0, m_0)$, where $m_0 = c_0^{1/2}(1 - m_{(2.3)})^{1/2}$, for $c_0 = c_{0(2.3)}$.

4.2.4. Let us estimate the function $\mathbf{V}_{(j,\,j+1)}(x,\,t)$ in the domain $\overline{G}_{(j,\,j+1)}$. Here, we need more smoothness of the components $\mathbf{U}(x,\,t)$, $\mathbf{V}_{(j)}(x,\,t)$ compared with the smoothness required for the estimates (4.7), (4.11). Assume that the data of the boundary value problem satisfy the condition (3.6), where

$$l \ge K - 2, \quad l_1 \ge 2\,K - 1; \tag{4.12}$$

this condition is stronger than the condition (4.6).

Taking into account estimates similar to (4.11), for the solution of the problem (4.10), we find the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}\,\mathbf{V}_{(j,\,j+1)}(x,\,t)\right| \le\; M\,\varepsilon^{-k}\,\exp\Big(-m\,\varepsilon^{-1}\,r\big(x,\Gamma_j\bigcap\Gamma_{j+1}\big)\Big),$$

$$(x,\,t)\in\overline{G},\quad k+2\,k_0\le K,\quad j=1,2,3,4,\quad m=m_{(4.11)}. \qquad (4.13)$$

4.3 Let us give a decomposition of the solution of the boundary value problem that will be used in the following construction.

The solution of the boundary value problem (2.2), (2.1) can be represented as a sum similar to (4.1):

$$\mathbf{u}(x,\,t)=\mathbf{U}^d(x,\,t)+\mathbf{V}^d(x,\,t),\quad (x,\,t)\in\overline{G}, \qquad (4.14)$$

where $\mathbf{U}^d_{(4.14)}(x,\,t)=\mathbf{U}_{(4.1)}(x,\,t)+\mathbf{U}^*(x,\,t),\quad \mathbf{V}^d_{(4.14)}(x,\,t)=\mathbf{V}^d_{(4.8)}(x,\,t).$

The additional component $\mathbf{U}^*(x,\,t)$ appears because the components of the function $\mathbf{V}^d_{(4.14)}(x,\,t)$ were constructed as solutions of boundary value problems in domains that are extensions of $\overline{G}$. For the function $\mathbf{U}^*(x,\,t)$, $(x,\,t)\in\overline{G}$, in the case of condition (4.12), we have the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}\,\mathbf{U}^*(x,\,t)\right| \le\; M,\quad (x,\,t)\in\overline{G},\quad k+2\,k_0\le K. \qquad (4.15)$$

For the component $\mathbf{U}^d_{(4.14)}(x,\,t)$ in the decomposition (4.14), taking into account the estimates (4.7), (4.15), we obtain

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}\,\mathbf{U}^d(x,\,t)\right| \;\le\; M\,\left[1+\varepsilon^{K-k-2}\right],\quad (x,\,t)\in\overline{G}, \qquad (4.16)$$

$$k+2\,k_0\le K.$$

For the components of the singular component $\mathbf{V}^d(x,\,t)$ in the decomposition (4.8), the estimates (4.11), (4.13) hold.

Theorem 1. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the conditions* (3.6), (4.12), *where* $K\ge 4$. *Then the solution components* $\mathbf{V}_{(j)}(x,\,t)$, $\mathbf{V}_{(j,\,j+1)}(x,\,t)$, *for* $j=1,2,3,4$, *and* $\mathbf{U}^d(x,\,t)$ *in the decompositions* (4.8), (4.14) *satisfy the estimates* (4.11), (4.13) *and* (4.16).

5 Finite Difference Scheme

5.1. When constructing a finite difference scheme for the problem (2.2), (2.1), we use a classical finite difference approximation on rectangular meshes (see, e.g., [8]). On the set $\overline{G}$ we introduce the grid

$$\overline{G}_h=\overline{D}_h\times\overline{\omega}_0,\quad \overline{D}_h=\overline{\omega}_1\times\overline{\omega}_2. \qquad (5.1)$$

Here $\overline{\omega}_s$ and $\overline{\omega}_0$ are, in general, arbitrary nonuniform meshes on the intervals $[0,\, d_s]$ and $[0,\, T]$ respectively. Set $h_s^i = x_s^{i+1} - x_s^i,\ x_s^i,\, x_s^{i+1} \in \overline{\omega}_s,\ h_s = \max_i h_s^i$, $h = \max_s h_s,\ s = 1,2$, and $h_t^k = t^{k+1} - t^k,\ t^k,\, t^{k+1} \in \overline{\omega}_0,\ h_t = \max_k h_t^k$. Assume that the condition $h \le MN^{-1},\ h_t \le MN_0^{-1}$ is fulfilled, where $N = \min_s N_s,\ s = 1,2,\ N_s + 1$ and $N_0 + 1$ are the number of nodes in the meshes $\overline{\omega}_s$ and $\overline{\omega}_0$, respectively.

To solve the problem, we use the implicit scheme on the grid $\overline{G}_h$:

$$\Lambda\, \mathbf{z}(x,\,t) = \mathbf{f}(x,\,t), \quad (x,\,t) \in G_h, \quad \mathbf{z}(x,\,t) = \boldsymbol{\varphi}(x,\,t), \quad (x,\,t) \in S_h. \qquad (5.2)$$

Here
$$G_h = G \bigcap \overline{G}_h, \quad S_h = S \bigcap \overline{G}_h,$$

$$\Lambda\, \mathbf{z}(x,\,t) \equiv \varepsilon^2\, \Lambda_2\, \mathbf{z}(x,\,t) - C(x,\,t)\, \mathbf{z}(x,\,t) - P(x,\,t)\, \delta_{\overline{t}}\, \mathbf{z}(x,\,t),$$

$$\Lambda_2 = \begin{pmatrix} \Lambda_2^1 & 0 \\ 0 & \Lambda_2^1 \end{pmatrix}, \quad \Lambda_2^1 = \sum_{s=1,2} \delta_{\overline{xs}\widehat{xs}}, \quad \mathbf{z}(x,\,t) = \left(z^1(x,\,t),\, z^2(x,\,t)\right)^T,$$

$(x,\,t) \in \overline{G}_h$. Here $\delta_{\overline{xs}\widehat{xs}}\, v(x,\,t) v_{\overline{xs}\widehat{xs}}(x,\,t)$, for $s = 1,2$, and $\delta_{\overline{t}}\, v(x,\,t)$ are the central second- and the backward first-order difference derivatives on nonuniform meshes [8].

5.2. We study the convergence of the scheme (5.2), (5.1) using the maximum principle [8] and under the assumption that the solution of the boundary value problem (2.2), (2.1) satisfies the estimates of Theorem 1.

Note that the operators

$$\Lambda^{(i)}_{(5.3)} \equiv \varepsilon^2\, \Lambda_2^1 - c^{ii}(x,\,t) - p^i(x,\,t)\, \delta_{\overline{t}}, \quad (x,\,t) \in \overline{G}_h, \quad i = 1,2, \qquad (5.3)$$

that approximate the operators $L^{(i)}_{(2.4)}$, are monotone [8].

Let us construct a special finite difference scheme for the problem (2.2), (2.1). On the set $\overline{G}$ we introduce the mesh

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad \overline{D}_h = \overline{D}_h^S = \overline{\omega}_1^S \times \overline{\omega}_2^S, \qquad (5.4)$$

where $\overline{\omega}_0$ is a uniform mesh on the interval $[0,\,T]$, and $\overline{\omega}_s^S = \overline{\omega}_s^S(\sigma_s)$ is a piecewise uniform mesh on the interval $[0, d_s]$. The mesh step-sizes of $\overline{\omega}_s^S$ are constant on the sets $[0, \sigma_s]$, $[d_s - \sigma_s, d_s]$ and $[\sigma_s, d_s - \sigma_s]$, and are equal, respectively to $h_s^{(1)} = 4\sigma_s N_s^{-1}$ and $h_s^{(2)} = 2(d_s - 2\sigma_s)N_s^{-1}$. The value σ_s is defined by

$$\sigma_s = \sigma_s(\varepsilon, N_s) = \min\left[\, 4^{-1} d_s,\ M\varepsilon \ln N_s \right], \quad s = 1,2, \quad M = 2m^{-1}_{(4.11)}.$$

To solve the problem (2.2), (2.1), we use the difference scheme (5.2) on the grid (5.4). Taking into account the estimates of Theorem 1, we establish the ε-uniform convergence of this scheme

$$|\, \mathbf{u}(x,\,t) - \mathbf{z}(x,\,t)\, | \ \le\ M\left[N^{-2} \ln^2 N + N_0^{-1}\right], \quad (x,\,t) \in \overline{G}_h. \qquad (5.5)$$

Theorem 2. *Let the components in the decompositions* (4.8)*,* (4.14) *of the solution of the boundary value problem* (2.2)*,* (2.1) *satisfy the estimates of Theorem 1 for* $K = 4$*. Then the solution of the difference scheme* (5.2)*,* (5.4) *converges to the solution of the boundary value problem* ε*-uniformly. The discrete solution satisfies the estimate* (5.5)*.*

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