Universality Theorems for Some Composite Functions

Kęstutis Janulis\(^a\), Donatas Jurgaitis\(^b\),
Antanas Laurinčikas\(^a\) and Renata Macaitienė\(^c\)

\(^a\)Faculty of Mathematics and Informatics, Vilnius University
Naugarduko str. 24, LT-03225 Vilnius, Lithuania

\(^b\)Institute of Informatics, Mathematics and E-Studies, Šiauliai University
P. Višinskio str. 19, LT-77156 Šiauliai, Lithuania

\(^c\)Research Institute, Šiauliai University
Vilniaus str. 88, LT-76285 Šiauliai, Lithuania

E-mail: kestutis.janulis@gmail.com
E-mail: d.jurgaitis@cr.su.lt
E-mail: antanas.laurincikas@mif.vu.lt
E-mail (corresp.): renata.macaitiene@mi.su.lt

Received June 15, 2015; revised November 17, 2015; published online January 15, 2016

Abstract. In [5], it was proved that a collection consisting from Dirichlet L-functions and periodic Hurwitz zeta-functions is universal in the sense that the shifts of those functions approximate simultaneously a given collection of analytic functions. In the paper, we prove theorems on the universality of composite functions of the above collection.

Keywords: Dirichlet L-function, Hurwitz zeta-function, mixed joint universality, periodic Hurwitz zeta-function, universality.

AMS Subject Classification: 11M06; 11M41.

1 Introduction

In [16], Voronin discovered the universality property of the Riemann zeta-function \(\zeta(s)\), \(s = \sigma + it\), on the approximation of analytic functions from a wide class by shifts \(\zeta(s + i\tau)\), \(\tau \in \mathbb{R}\). At the moment, it is known that the majority of zeta and L-functions are universal in the above sense. Also, some zeta and L-functions are jointly universal: their shifts approximate simultaneously a given collection of analytic functions. A series of works are devoted to mixed joint universality when a collection of analytic functions are approximated simultaneously by shifts of zeta-functions with Euler product and without Euler product. The first result in this direction belongs to H. Mishou who proved [14] the joint universality of the function \(\zeta(s)\) and the Hurwitz zeta-function \(\zeta(s, \alpha)\)
with transcendental parameter $\alpha$. This result has been generalized in [6] for a periodic zeta and a periodic Hurwitz zeta-functions. In [7], the mixed joint universality has been obtained for a wide collection consisting from periodic zeta and periodic Hurwitz zeta-functions. We remind that the periodic Hurwitz zeta-function $\zeta(s, \alpha; a)$, where $0 < \alpha \leq 1$, is a fixed parameter and $a = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ is a periodic sequence of complex numbers, is a generalization of the classical Hurwitz zeta-function $\zeta(s, \alpha)$ when $a_m \equiv 1$, and is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. In [2], the mixed joint universality for a system of functions

$$\zeta(s), \zeta(s, \alpha_1, a_{11}), \ldots, \zeta(s, \alpha_1, a_{1l_1}), \ldots, \zeta(s, \alpha_r, a_{rl}), \ldots, \zeta(s, \alpha_r, a_{rl_r})$$

has been considered. In a series of papers [11,12,15], the Riemann zeta-function has been replaced by zeta-functions of certain cusp forms. In [5], in place of the function $\zeta(s)$ a collection of Dirichlet $L$-functions $L(s, \chi)$ has been put. We will state the latter result.

Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $K$ the class of compact subsets of the strip $D$ with connected complements, and by $H_0(K)$ and $H(K)$, $K \in \mathcal{K}$, the classes of continuous non-vanishing and continuous on $K$ functions, respectively, which are analytic in the interior of $K$. Let $\text{meas}_A$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Suppose that $a_{jl} = \{a_{m_{jl}} : m \in \mathbb{N}_0\}$ is a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$. Let $k_j$ be the least common multiple of the periods $k_{j1}, \ldots, k_{jl_j}$, and

$$A_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \cdots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \cdots & a_{2jl_j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{kj1} & a_{kj2} & \cdots & a_{kj{l_j}} \end{pmatrix}, \quad j = 1, \ldots, r.$$ 

Then, in [5], the following theorem has been proved.

**Theorem 1.** Suppose that $\chi_1, \ldots, \chi_d$ are pairwise non-equivalent Dirichlet characters, the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$, and that $\text{rank}(A_j) = l_j$, $j = 1, \ldots, r$. For $j = 1, \ldots, d$, let $K_j \in \mathcal{K}$ and $f_j \in H_0(K_j)$, and, for $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, let $K_{jl} \in \mathcal{K}$ and $f_{jl} \in H(K_{jl})$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq d} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon, \quad \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; a_{jl}) - f_{jl}(s)| < \varepsilon \right\} > 0.$$
Denote by $H(D)$ the space of analytic on $D$ functions equipped with the topology of uniform convergence on compacta. In [8] and [10], the Voronin theorem has been generalized for $F(\zeta(s), \zeta(s, \alpha))$ with certain operators $F: H(D) \to H(D)$, in [9], the universality of $F(\zeta(s), \zeta(s, \alpha))$ has been studied with operators $F: H^2(D) \to H(D)$. The papers [3] and [4] are devoted to the universality of the functions $F(L(s, \chi_1), \ldots, L(s, \chi_d), \zeta(s, \alpha_1), \ldots, \zeta(s, \alpha_r))$ for some operators $F: H^{(r_1+r_2)}(D) \to H(D)$. The aim of the present paper is the universality of composite functions of a collection of $L$ and zeta-functions in Theorem 1, i.e., we consider the universality of the functions

$$F\left(L(s, \chi_1), \ldots, L(s, \chi_d), \zeta(s, \alpha_1; a_{11}), \ldots, \zeta(s, \alpha_r; a_{rl})\right)$$

for some operators $F$.

First we deal with approximation of functions from the class $H(K)$, $K \in \mathcal{K}$. Let, for brevity, $v = d + l_1 + \cdots + l_r$. We say that the operator $F: H^v(D) \to H(D)$ belongs to the class $\text{Lip}(\beta_1, \ldots, \beta_v)$, $\beta_1 > 0, \ldots, \beta_v > 0$, if the following hypotheses are satisfied:

1° For every polynomial $p = p(s)$ and all sets $K_1, \ldots, K_d \in \mathcal{K}$, there exists an element $g = (g_1, \ldots, g_d, g_{11}, \ldots, g_{1l_1}, \ldots, g_{rl_1}, \ldots, g_{rl_r}) \in F^{-1}\{p\} \subset H^v(D)$ such that $g_j \neq 0$ on $K_j$, $j = 1, \ldots, d$;

2° For all $K \in \mathcal{K}$, there exist a constant $c > 0$ and sets $K_1, \ldots, K_v \in \mathcal{K}$ such that, for all $(g_1, \ldots, g_v) \in H^v(D)$, $j = 1, 2$,

$$\sup_{s \in K} |F(g_{11}(s), \ldots, g_{1v}(s)) - F(g_{21}(s), \ldots, g_{2v}(s))| \leq c \sup_{1 \leq j \leq v} \sup_{s \in K_j} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}.$$

**Theorem 2.** Suppose that $\chi_1, \ldots, \chi_d$ are pairwise non-equivalent Dirichlet characters, the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$, $\text{rank}(A_j) = l_j$, $j = 1, \ldots, r$, and that $F \in \text{Lip}(\beta_1, \ldots, \beta_v)$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,

$$\lim_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |F(L(s + i\tau, \chi_1), \ldots, L(s + i\tau, \chi_d), \zeta(s + i\tau, \alpha_1; a_{11}), \ldots, \zeta(s + i\tau, \alpha_r; a_{rl}) - f(s)| < \varepsilon \right\} > 0.$$
difficult to see that, for each polynomial $p = p(s)$ and all sets $K_1, \ldots, K_d \in \mathcal{K}$, there exists an element $g \in F^{-1}\{p\}$ such that $g_j(s) \neq 0$ on $K_j$, $j = 1, \ldots, d$. For example, if

$$p(s) = a_k s^k + a_{k-1} s^{k-1} + \cdots + a_0, \quad a_k \neq 0,$$

we can take $g = (1, \ldots, 1, 1, \ldots, 1, \ldots, g_{rl}, \ldots, g_{rl})$, where

$$g_{rl}(s) = \frac{1}{c_{rl}} \left( \frac{a_0 s^{n_{rl}}}{(k+1) \cdots (k+n_{rl})} + \cdots + \frac{a_0 s^{n_{rl}}}{1 \cdots n_{rl}} \right).$$

Thus, hypothesis 1° of the class $\text{Lip}(\beta_1, \ldots, \beta_v)$ is satisfied.

Hypothesis 2° of the class $\text{Lip}(\beta_1, \ldots, \beta_v)$ follows from the Cauchy integral formula. We write $F$ in a more convenient form

$$F(g_1, \ldots, g_v) = \sum_{j=1}^v c_j g_j^{(n_j)}.$$

Let $K \in \mathcal{K}$, and $K \subset G \subset K_1$, where $G$ is an open set and $K_1 \in \mathcal{K}$. Moreover, let $L$ be a simple closed contour lying in $K_1 \setminus G$ and containing inside the $K$. Then the Cauchy integral formula shows that, for $(g_{j1}, \ldots, g_{jv}) \in H^v(D)$, $j = 1, 2$, and $s \in K$,

$$|F(g_{j1}(s), \ldots, g_{jv}(s)) - F(g_{21}(s), \ldots, g_{2v}(s))| = \left| \sum_{j=1}^v c_j \frac{n_{j1}!}{2\pi i} \int_L \frac{g_{1j}(z) - g_{2j}(z)}{(z - s)^{n_{j1}+1}} \, dz \right|$$

$$\leq \sum_{j=1}^v |c_j| |c_j| \sup_{s \in L} |g_{1j}(s) - g_{2j}(s)| \leq c \sup_{1 \leq j \leq v} \sup_{s \in K_1} |g_{1j}(s) - g_{2j}(s)|$$

with some constants $C_j > 0$, $j = 1, \ldots, v$, and $c > 0$. Thus we have that $F \in \text{Lip}(1, \ldots, 1)$, and in this case, $K_1 = \cdots = K_v = K_1$.

Now we give some other classes of operators $F$. Let

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Moreover, $v_1 = \sum_{j=1}^v l_j$.

**Theorem 3.** Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$, and the sequences $a_l, j = 1, \ldots, r, l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, and that $F : H^v(D) \to H(D)$ be a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S^d \times H^{v_1}(D))$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 2 is true.

We note that the hypothesis $(F^{-1}G) \cap (S^d \times H^{v_1}(D)) \neq \emptyset$ for every open set $G \subset H(D)$ is general but sufficiently complicated. Obviously, it is satisfied if every $g \in H(D)$ has a preimage in the set $S^d \times H^{v_1}(D)$. On the other hand, Theorem 3 implies the following modification of Theorem 2.

**Theorem 4.** Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$, and the sequences $a_l, j = 1, \ldots, r, l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, and that $F : H^v(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D))$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 2 is true.
Clearly, hypothesis $2^\circ$ of the class $\text{Lip}(\beta_1, \ldots, \beta_v)$ implies the continuity of $F$. However, hypothesis $1^\circ$ is weaker than the requirement $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D)) \neq \emptyset$.

Non-vanishing of the polynomial $p(s)$ in a bounded region can be controlled by its constant term. Therefore, sometimes it is more convenient to consider operators $F$ on the space $H^v(D_V, D) = H^d(D_V) \times H^{v_1}(D)$, where, for $V > 0$, $D_V = \{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V \}$. Analogically, let

$$S_V = \{ g \in H(D_V) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.$$ 

Then we have the following result.

**Theorem 5.** Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$, and the sequences $a_{jl}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, $K \subset K$, $f(s) \in H(K)$ and $V > 0$ is such that $K \subset D_V$. Let $F : H^v(D_V, D) \to H(D_V)$ be a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D))$ is not empty. Then the assertion of Theorem 2 is true.

For example, Theorem 5 can be applied for the operator

$$F(g_1, \ldots, g_v) = c_1g_1^{(n_1)} + \cdots + c_dg_d^{(n_d)}, \quad n_1, \ldots, n_d \in \mathbb{N}.$$ 

Now we consider approximation of analytic functions from the image of the set $S^d \times H^{v_1}(S)$ of the operator $F : H^v(D) \to H(D)$.

**Theorem 6.** Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$, and the sequences $a_{jl}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, and that $F : H^v(D) \to H(D)$ is a continuous operator. Let $K \subset D$ be a compact subset, and $f(s) \in F(S^d \times H^{v_1}(D))$. Then the assertion of Theorem 2 is true.

It is not easy to describe the set $F(S^d \times H^{v_1}(D))$. The next theorem is an example with sufficiently simple set contained in $F(S^d \times H^{v_1}(D))$.

Suppose that $a_1, \ldots, a_k \in \mathbb{C}$ are pairwise different numbers, and

$$H_k(D) = \{ g \in H(D) : (g(s) - a_j)^{-1} \in H(D), \ j = 1, \ldots, k \}.$$ 

**Theorem 7.** Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$, and the sequences $a_{jl}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, and that $F : H^v(D) \to H(D)$ is a continuous operator such that $F(S^d \times H^{v_1}(D)) \supset H_k(D)$. For $k = 1$, let $K \subset K$, $f(s) \in H(K)$ and $f(s) \not\equiv a_1$ on $K$. For $k \geq 2$, let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_k(D)$. Then the assertion of Theorem 2 is true.

For example, let $k = 2$ and $a_1 = 1$, $a_2 = -1$. Then Theorem 7 implies the universality of the function

$$\sin \left( L(s, \chi_1) + \cdots + L(s, \chi_d) + \zeta(s, \alpha_1; a_{11}) + \cdots + \zeta(s, \alpha_1; a_{1l_1}) + \cdots + \zeta(s, \alpha_r; a_{r1}) + \cdots + \zeta(s, \alpha_r; a_{rl_r}) \right).$$
For this, it suffices to consider the equation
\[ \frac{e^{i\Sigma(s)} - e^{-i\Sigma(s)}}{2i} = f, \quad f \in H(D), \quad a_1 = 1, \quad a_2 = -1, \]
where \( \Sigma(s) \) is the sum under the sign of sin.

### 2 Proof of Theorem 2

Theorem 2 is a result of Theorem 1, properties of the class \( \text{Lip}(\beta_1, \ldots, \beta_v) \) and of the Mergelyan theorem on the approximation of analytic functions by polynomials. We state this theorem in the form of the next lemma.

**Lemma 1.** Suppose that \( K \subset \mathbb{C} \) is a compact subset with connected complement, and \( f(s) \) is a continuous function on \( K \) which is analytic in the interior of \( K \). Then, for every \( \varepsilon > 0 \), there exists a polynomial \( p(s) \) such that
\[
\sup_{s \in K} |f(s) - p(s)| < \varepsilon.
\]

Proof of the lemma can be found in [13] and [17].

**Proof of Theorem 2.** Lemma 1 implies the existence of the polynomial \( p = p(s) \) such that
\[
\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \quad (2.1)
\]

Using hypothesis 1° of the class \( \text{Lip}(\beta_1, \ldots, \beta_v) \), we have that, for all sets \( K_1, \ldots, K_d \in \mathcal{K} \), there exists an element \((g_1, \ldots, g_d, g_{11}, \ldots, g_{1l_1}, \ldots, g_{r1}, \ldots, g_{rl_r}) \in F^{-1}\{p\}\) such that \( g_j(s) \neq 0 \) on \( K_j, \quad j = 1, \ldots, d \). Suppose that \( \tau \in \mathbb{R} \) satisfies the inequalities
\[
\sup_{1 \leq j \leq d s \in K_j} \sup_{1 \leq j \leq d s \in K_j} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < c^{-\frac{1}{2}} \left( \frac{\varepsilon}{4} \right)^{\frac{1}{\beta}}, \quad (2.2)
\]
\[
\sup_{1 \leq j \leq r \leq 1 \leq l \leq l_j s \in K_{jl}} ||\zeta(s + i\tau, \alpha_j; a_{jl}) - f_{jl}(s)|| < c^{-\frac{1}{2}} \left( \frac{\varepsilon}{4} \right)^{\frac{1}{\beta}}, \quad (2.3)
\]
where the sets \( K_1, \ldots, K_d, K_{d1}, \ldots, K_{d1_l}, \ldots, K_{rl}, \ldots, K_{rl_r} \in \mathcal{K} \) correspond the set \( K \) in hypothesis 2° of the class \( \text{Lip}(\beta_1, \ldots, \beta_v) \), and \( \beta = \min_{1 \leq j \leq v} \beta_j \), with notation \( K_{jl} = K_{d1_l+ \cdots + l_{r-1} + l}, \quad l = 1, \ldots, l_r \). Then, in view of Theorem 1, the set of \( \tau \) satisfying inequalities (2.2) and (2.3) has a positive lower density. Moreover, hypothesis 2° of the class \( \text{Lip}(\beta_1, \ldots, \beta_v) \) shows that, for such \( \tau \),
\[
\sup_{s \in K} |F(L(s + i\tau, \chi_j; a)) - p(s)| \leq \sup_{1 \leq j \leq d s \in K_j} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)|^{\beta_j} + c \sup_{1 \leq j \leq r \leq 1 \leq l \leq l_j s \in K_{jl}} ||\zeta(s + i\tau, \alpha_j; a_{jl}) - f_{jl}(s)||^{\beta_{jl}} \leq 2cc^{-\frac{\theta}{2}} \left( \frac{\varepsilon}{4} \right)^{\frac{\theta}{2}} = \frac{\varepsilon}{2}. \quad (2.4)
\]
Here $\chi = (\chi_1, \ldots, \chi_d)$, $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\mathbf{a} = (a_{11}, \ldots, a_{1l_1}, \ldots, a_{r1}, \ldots, a_{rl_r})$ and

$$L(s + i\tau, \chi, \alpha, \mathbf{a}) = (L(s, \chi_1), \ldots, L(s, \chi_d), \zeta(s, \alpha_1; a_{11}), \ldots, \zeta(s, \alpha_1; a_{1l_1}), \ldots, \zeta(s, \alpha_r; a_{r1}), \ldots, \zeta(s, \alpha_r; a_{rl_r})),$$

and $\beta_{1l} = \beta_{d+l}$, $l = 1, \ldots, l_1$, \ldots, $\beta_{rl} = \beta_{d+l_1+\ldots+l_{r-1}+l}$, $l = 1, \ldots, l_r$. Thus, by the above remark,

$$\liminf_{T \to \infty} \frac{1}{T} \frac{1}{\text{meas}} \left\{ \tau \in [0, T] : \sup_{s \in K} \left| F(L(s + i\tau, \chi, \alpha, \mathbf{a})) - p(s) \right| < \frac{\epsilon}{2} \right\} > 0.$$

Combining this with inequality (2.1) proves the theorem.

### 3 Elements of probability theory

For the proof of Theorems 3 – 7, we apply a probabilistic approach based on limit theorems for weakly convergent probability measures in the space of analytic functions. We start with a limit theorem for $L(s + i\tau, \chi, \alpha, \mathbf{a})$ obtained in [5], Theorem 2.

Denote by $B(X)$ the Borel $\sigma$-field of the space $X$. Let $\gamma = \{ s \in \mathbb{C} : |s| = 1 \}$ be the unit circle on the complex plane, and

$$\Omega = \prod_{p \in \mathcal{P}} \gamma_p, \quad \hat{\Omega} = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\mathcal{P}$ is the set of all prime numbers, and $\gamma_p = \gamma$ for all $p \in \mathcal{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. In view of the classical Tikhonov theorem, the tori $\Omega$ and $\hat{\Omega}$ with the product topology and pointwise multiplication are compact topological Abelian groups. Moreover, let

$$\hat{\Omega} = \Omega \times \hat{\Omega}_1 \times \cdots \times \hat{\Omega}_r,$$

where $\hat{\Omega}_j = \hat{\Omega}$ for all $j = 1, \ldots, r$. Then again $\hat{\Omega}$ is a compact topological Abelian group. This leads to the probability space $(\Omega, \mathcal{B}(\Omega), \mu_H)$, where $\mu_H$ is the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Denote by $\omega(p)$ the projection of the element $\omega \in \Omega$ to the coordinate space $\gamma_p$, $p \in \mathcal{P}$, and by $\hat{\omega}_j(m)$ the projection of an element $\hat{\omega}_j \in \hat{\Omega}_j$ to the coordinate space $\gamma_m$, $m \in \mathbb{N}_0$, $j = 1, \ldots, r$. Let $p^k|m$ mean that $p^k | m$ but $p^{k+1} \nmid m$. Extend the function $\omega(p)$ to the set $\mathbb{N}$ by taking

$$\omega(m) = \prod_{p^k|m} \omega^k(m), \quad m \in \mathbb{N}.$$

Denote by $\omega = (\omega, \hat{\omega}_1, \ldots, \hat{\omega}_r)$ the elements of $\hat{\Omega}$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), \mu_H)$, define the $H^u(D)$-valued random element $L(s, \chi, \alpha, \omega, \mathbf{a})$ by the formula

$$L(s, \chi, \alpha, \omega, \mathbf{a}) = (L(s, \omega, \chi_1), \ldots, L(s, \omega, \chi_d), \zeta(s, \alpha_1, \omega_1; a_{11}), \ldots, \zeta(s, \alpha_1, \omega_1; a_{1l_1}), \ldots, \zeta(s, \alpha_r, \omega_r; a_{r1}), \ldots, \zeta(s, \alpha_r, \omega_r; a_{rl_r})).$$

where
\[ L(s, \omega, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)\omega(m)}{m^s}, \quad j = 1, \ldots, d, \]
and
\[ \zeta(s, \alpha_j, \omega_j; a_{jl}) = \sum_{m=1}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j. \]

We note that the latter series are uniformly convergent on compact subsets of \( D \) for almost all \( \omega \in \Omega \). Moreover, for almost \( \omega \in \Omega \), \( L(s, \omega, \chi_j) \) can be written in the form
\[ L(s, \omega, \chi_j) = \prod_p \left( 1 - \frac{\chi_j(p)\omega(p)}{p^s} \right)^{-1}. \]

Denote by \( P_L \) the distribution of the random element \( L(s, \chi, \alpha, \omega, a) \), i.e., the probability measure
\[ P_L(A) = \mathbb{m}_H(\omega \in \Omega : L(s, \chi, \alpha, \omega, a) \in A), \quad A \in \mathcal{B}(H^v(D)). \]

Then we have the following limit theorem [5].

**Lemma 2.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \). Then
\[ P_T(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas} \{ \tau \in [0, T] : L(s + i\tau, \chi, \alpha, \omega, a) \in A \}, \quad A \in \mathcal{B}(H^v(D)) \]
converges weakly to \( P_L \) as \( T \to \infty \).

For the proof a limit theorem for composite function \( F(L(s, \chi, \alpha, \omega, a)) \), we will apply an assertion on the preservation of the weak convergence under mappings. Let \( X_1 \) and \( X_2 \) be two metric spaces, and let \( u : X_1 \to X_2 \) be a \((\mathcal{B}(X_1), \mathcal{B}(X_2))\)-measurable mapping, i.e.,
\[ u^{-1}\mathcal{B}(X_2) \subset \mathcal{B}(X_1). \]

Then every probability measure \( P \) on \( (X_1, \mathcal{B}(X_1)) \) induces the unique probability measure \( Pu^{-1} \) on \( (X_2, \mathcal{B}(X_2)) \) defined by
\[ Pu^{-1}(A) = P(u^{-1}A), \quad A \in \mathcal{B}(X_2). \]

It is well known that the continuity of \( u \) implies its \((\mathcal{B}(X_1), \mathcal{B}(X_2))\)-measurability.

**Lemma 3.** Suppose that \( P_n \) converges weakly to \( P \) as \( n \to \infty \), and that the mapping \( u : X_1 \to X_2 \) is continuous. Then \( P_nu^{-1} \) converges weakly to \( Pu^{-1} \) as \( n \to \infty \).

Proof of the lemma is given in [1].
Lemma 4. Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$ and that the operator $F : H^v(D) \to H(D)$ is continuous. Then

\[
P_{T,F}(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\left\{ \tau \in [0,T] : F(L(s+i\tau, \chi, \alpha, a)) \in A \right\}, \quad A \in \mathcal{B}(H^v(D)),
\]

converges weakly to $P_LF^{-1}$ as $T \to \infty$.

Proof. The lemma is an immediate consequence of Lemmas 2 and 3. \(\Box\)

For the proof of universality theorems for $F(L(s+i\tau, \chi, \alpha, a))$, we also need the explicit form of the support of the measure $P_LF^{-1}$. We apply a result of [5] on the support of the measure $P_L$. 

Lemma 5. Suppose that $\chi_1, \ldots, \chi_d$ are pairwise non-equivalent Dirichlet characters, the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $\text{rank}(A_j) = l_j, j = 1, \ldots, r$. Then the support of $P_L$ is the set $S^d \times H^v(D)$.

Lemma 6. Suppose that the hypotheses of Lemma 5 are satisfied, and that the operator $F : H^v(D) \to H(D)$ is continuous. Then the support of $P_LF^{-1}$ is the closure of the set $F(S^d \times H^v(D))$.

Proof. Let $g$ be an arbitrary element of the set $F(S^d \times H^v(D))$, and $G$ be any open neighbourhood of $g$. Then $F^{-1}G$ is an open neighbourhood of a certain element of the set $S^d \times H^v(D)$. Therefore, Lemma 5 and properties of a support imply that $P_L(F^{-1}G) > 0$, hence, $P_LF^{-1}(G) > 0$. Moreover, in virtue of Lemma 5 again,

\[
P_LF^{-1}(F(S^d \times H^v(D))) = P_L(S^d \times H^v(D)) = 1.
\]

Since the support of $P_LF^{-1}$ is a closed set, from this the lemma follows. \(\Box\)

We also need one equivalent of the weak convergence of probability measures.

Lemma 7. Let $P_n, n \in \mathbb{N}$, and $P$ be probability measures on $(X, \mathcal{B}(X))$. Then, $P_n$, as $n \to \infty$, converges weakly to $P$ if and only if, for every open set $G \subset X$,

\[
\liminf_{n \to \infty} P_n(G) \geq P(G).
\]

The lemma is a part of Theorem 2.1 from [1].

4 Proofs of other universality theorems

Proof of Theorem 3. It is not difficult to see that, under hypotheses of Theorem 3, the support of the measure $P_LF^{-1}$ is the whole of $H(D)$. Indeed, if $(F^{-1}G) \cap (S^d \times H^v(D)) \neq \emptyset$ for every open set $G \subset H(D)$, then we have that, for every element $g \in H(D)$ and its open neighbourhood $G$, there exists an element of the set $F(S^d \times H^v(D))$ which belongs to the set $G$. This shows
that the set $F(S^d \times H^{v_1}(D))$ is dense in $H(D)$. Since, by Lemma 6, the support of $P_L F^{-1}$ is the closure of $F(S^d \times H^{v_1}(D))$, from this we obtain that the support of $P_L F^{-1}$ is the whole of $H(D)$.

In view of Lemma 1, there exists a polynomial $p(s)$ satisfying inequality (2.1). Define the set

$$G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$  

Obviously, $G$ is an open neighbourhood of $p(s)$ which is an element of the support of $P_L F^{-1}$. Therefore, $P_L F^{-1}(G) > 0$, and Lemmas 4 and 7 imply the inequality

$$\lim \inf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : F(L(s + i\tau, \chi, \alpha, a)) \in G \right\} > 0,$$

or, by definition of $G$,

$$\lim \inf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |F(L(s + i\tau, \chi, \alpha, a)) - p(s)| < \frac{\varepsilon}{2} \right\} > 0.$$  

Combining this with (2.1) gives the assertion of the theorem.

**Proof of Theorem 4.** Let $\{K_l : l \in \mathbb{N}\} \subset D$ be the sequence of compact subsets such that $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$,

$$D = \bigcup_{l=1}^{\infty} K_l,$$

and, for every compact subset $K \subset D$, there exists $K_l$ such that $K \subset K_l$. For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \sup_{s \in K_l} \frac{|g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$  

Then $\rho$ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta. Moreover, from the definition of $\rho$ we have that the function $g_2$ approximates $g_1$ with suitable accuracy if $g_2$ approximate $g_1$ uniformly on $K_l$ for sufficiently large $l$. Obviously, we may choose the sets $K_l$ with connected complements, for example, we can take the closed rectangles. Thus, in $H(D)$, we can limit ourselves by approximation of functions on compact subsets with connected complements.

We will show that the hypotheses of the theorem imply those of Theorem 3. Let $G \subset H(D)$ be an arbitrary non-empty open set. Then, in view of Lemma 1 and the above remark on approximation in $H(D)$, there exists a polynomial $p(s) \in G$. Therefore, the hypothesis $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D)) \neq \emptyset$ implies that of Theorem 3 that the set $(F^{-1} G) \cap (S^d \times H^{v_1}(D))$ is non-empty. Thus, Theorem 4 is a corollary of Theorem 3.

**Proof of Theorem 5.** We apply the arguments used in the proof of Theorem 3 with obvious modifications.
Proof of Theorem 6. Since \( f(s) \in F(S^d \times H^{s_1}(D)) \), it follows from Lemma 6 that \( f(s) \) is an element of the support of the measure \( P_{L}F^{-1} \). Hence, \( P_{L}F^{-1}(G) > 0 \) for
\[
G = \left\{ g \in H(D) : \sup_{s \in K}|g(s) - p(s)| < \varepsilon \right\}.
\]

Therefore, the theorem is a consequence of Lemmas 4 and 7.

Proof of Theorem 7. First suppose that \( k = 1 \). By Lemma 1, there exists a polynomial \( p(s) \) such that
\[
\sup_{s \in K}|f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{4.1}
\]

Since \( f(s) \neq a_1 \) on \( K \), then also \( p(s) \neq a_1 \) on \( K \) if \( \varepsilon > 0 \) is rather small. Therefore, on \( K \) we can define a continuous branch of the logarithm \( \log(p(s) - a_1) \) which will be an analytic function in the interior of \( K \). Again by Lemma 1, we can find a polynomial \( q(s) \) such that
\[
\sup_{s \in K} \left| p(s) - a_1 - e^{q(s)} \right| < \frac{\varepsilon}{4}. \tag{4.2}
\]
Let, for brevity, \( g_{a_1}(s) = a_1 + e^{q(s)} \). Then, clearly, \( g_{a_1}(s) \in H(D) \), and \( g_{a_1}(s) \neq a_1 \) on \( D \). Thus, \( g_{a_1}(s) \in H_1(D) \). In view of Lemma 6, the support of the measure \( P_{L}F^{-1} \) contains the closure of the set \( H_1(D) \). Therefore, the function \( g_{a_1}(s) \) is an element of the support of the measure \( P_{L}F^{-1} \). Hence, \( P_{L}F^{-1}(G) > 0 \), where
\[
G = \left\{ g \in H(D) : \sup_{s \in K}|g(s) - g_{a_1}(s)| < \frac{\varepsilon}{2} \right\}.
\]

This together with Lemmas 4 and 7 shows that
\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left \{ \tau \in [0, T] : \sup_{s \in K}|F(L(s+i\tau, \chi, \alpha, a)) - g_{a_1}(s)| < \frac{\varepsilon}{2} \right \} > 0. \tag{4.3}
\]
Inequalities (4.1) and (4.2) imply that
\[
\sup_{s \in K}|f(s) - g_{a_1}(s)| < \frac{\varepsilon}{2}.
\]
This and (4.3) yield the assertion of the theorem in the case \( k = 1 \).

The case \( k \geq 2 \) is contained in Theorem 6.

References


