Difference Equations in a Multidimensional Space

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Received August 25, 2015; revised February 28, 2016; published online May 15, 2016

Abstract. One considers a general difference equation in a multidimensional space with continuous coefficients in the space of integrable functions. Necessary and sufficient conditions for a Fredholm property are obtained with a help of the Fourier transform and the Riemann boundary value problem. For simplest cases solvability conditions and formula of a general solution for the difference equation are given.

Keywords: difference equation, symbol, singular integral equation, Riemann boundary value problem, index.

AMS Subject Classification: 39A06; 35S05.

1 Introduction

We consider a general difference equation in a multidimensional space of the following type

\[
\sum_{|k|=0}^{+\infty} a_k(x)u(x + \beta_k) = v(x), \quad x \in D, \tag{1.1}
\]

where \(D\) is \(\mathbb{R}^m\) or \(\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x_1, \ldots, x_m), x_m > 0\}\), \(k\) is multi-index \(k = (k_1, \ldots, k_m), \beta_k = (\beta_{k_1}, \ldots, \beta_{k_m}) \in D\). Our main goal is to describe a Fredholm property for such equations. Such equations have variable coefficients, and only for one-dimensional equations on a straight line with constant coefficients one can construct exact solution [4,6,11]. We’ll use methods of the theory of singular integral equations [2, 3, 5, 7], boundary value problems [1], and the Wiener–Hopf technique [8]. Some results related to certain classes of difference equations are described in papers [12,13,15,16,17].

\* This work was partially supported by Russian Foundation for Basic Research and government of Lipetsk region of Russia, project no. 14-41-03595-a
This paper is devoted to equations with a continual variable. The case of a discrete variable will be considered in a separate paper because there are certain principal distinctions.

2 The Fourier transform and symbol

If we consider the equation with constant coefficients in the whole space $\mathbb{R}^m$
\begin{equation}
\sum_{|k|=0}^{+\infty} a_k u(x + \beta_k) = v(x), \quad x \in \mathbb{R}^m,
\end{equation}
then we can use the Fourier transform
\begin{equation}
\tilde{u}(\xi) = \int_{\mathbb{R}^m} e^{ix \cdot \xi} u(x) dx
\end{equation}
and obtain an equivalent equation in the space $L_2(\mathbb{R}^m)$
\begin{equation}
\sigma(\xi) \tilde{u}(\xi) = \tilde{v}(\xi),
\end{equation}
where
\begin{equation}
\sigma(\xi) = \sum_{k=0}^{+\infty} a_k e^{i\beta_k \cdot \xi}, \quad \xi \in \mathbb{R}^m.
\end{equation}

It implies necessary and sufficient condition for a unique solvability of the equation (2.1): if $\sigma \in L_\infty(\mathbb{R}^m)$ then this is
\begin{equation}
\text{ess inf}_{\xi \in \mathbb{R}^m} |\sigma(\xi)| > 0.
\end{equation}

Unfortunately we can’t use this approach if we are in the space $\mathbb{R}^m_+$ and consider the equation (1.1) with constant coefficients
\begin{equation}
\sum_{|k|=0}^{+\infty} a_k u(x + \beta_k) = v(x), \quad x \in \mathbb{R}^m_+,
\end{equation}
because we have no description for Fourier image of the space $L_2(\mathbb{R}^m_+)$. Hence the first step is to obtain such description.

This description is a very simple and it is described in [1]. The book [1] is devoted to constructing theory of pseudo differential equations on manifolds with a smooth boundary but methods introduced in the book are applicable to our situation also. Here we’ll give a brief sketch of main ideas from [1].

If $u \in L_2(\mathbb{R}^m)$ then
\begin{equation}
FP_\pm u = \Pi_\pm Fu,
\end{equation}
where $P_\pm$ is the projection operator on $\mathbb{R}^m_\pm$, i.e. $(P_\pm u)(x) = u(x)$ if $x \in \mathbb{R}^m_\pm$, and $(P_\pm u)(x) = 0$ otherwise, $\Pi_\pm$ is the following operator
\begin{equation}
(\Pi_\pm u)(\xi) = \frac{i}{2\pi \tau} \int_{-\infty}^{+\infty} \frac{u(x', \eta_m) d\eta_m}{\xi_m \pm i\tau - \eta_m}, \quad \xi' = (\xi_1, \cdots, \xi_{m-1}).
\end{equation}

This relation is a very important for constructing a solution of the difference equation (2.3).

Let’s introduce the following notations. Let \( A, B \) be difference operators of the type
\[
(Au)(x) = \sum_{|k|=0}^{+\infty} a_k u(x + \alpha_k), \quad (Bu)(x) = \sum_{|k|=0}^{+\infty} b_k u(x + \beta_k), \quad x \in \mathbb{R}^m.
\]

The functions
\[
\sigma_A(\xi) = \sum_{k=0}^{+\infty} e^{i\alpha_k \cdot \xi}, \quad \sigma_B(\xi) = \sum_{k=0}^{+\infty} e^{i\beta_k \cdot \xi}, \quad \xi \in \mathbb{R}^m
\]
are called the symbols of the operators \( A, B \) respectively.

Obviously these operators \( A, B \) are linear bounded operators \( L_2(\mathbb{R}^m) \rightarrow L_2(\mathbb{R}^m) \) if \( \sigma_A, \sigma_B \in L_\infty(\mathbb{R}^m) \).

The equation (2.3) can be written in an operator form as follows
\[
(P + Au)(x) = v(x), \quad x \in \mathbb{R}_+^m. \tag{2.4}
\]

One can consider also more general equation in the space \( L_2(\mathbb{R}^m) \)
\[
(AP_+ + BP_-)U = V \tag{2.5}
\]
and conclude that the equation (2.4) is equivalent to the equation (2.5) with \( B \equiv I \), where \( I \) is identity operator. This equivalence means that if we know the solution of (2.4) we can write the solution of
\[
(AP_+ + IP_-)U = V
\]
and vice versa. That’s why we’ll consider more general equation (2.5).

After applying the Fourier transform to (2.5) we obtain
\[
\sigma_A(\xi)(P_+ \tilde{U})(\xi) + \sigma_B(\xi)(P_- \tilde{U})(\xi) = \tilde{V}(\xi) \tag{2.6}
\]
and the last equation (2.6) is one-dimensional characteristic singular integral equation with the parameter \( \xi' \). Such conclusion we have because the introduced operators \( P_\pm \) are connected to the operator (Plemelj–Sokhotskii formulas)
\[
(Hu)(\xi) = \frac{1}{\pi i} v.p. \int_{-\infty}^{+\infty} \frac{u(\xi', \eta_m) d\eta_m}{\xi_m - \eta_m},
\]
which is well-known as the Hilbert transform or one-dimensional singular integral operator [2,3,5,7].

3 Singular integral equation and Riemann boundary value problem

The equation (2.6) can be rewritten as the Riemann boundary value problem
\[
(P_+ \tilde{U})(\xi', \xi_m) = -\sigma_A^{-1}(\xi', \xi_m)\sigma_B(\xi', \xi_m)(P_- \tilde{U})(\xi', \xi_m)
+ \sigma_A^{-1}(\xi', \xi_m)\tilde{V}(\xi', \xi_m), \quad \xi_m \in \mathbb{R},
\]
or as the one-dimensional singular integral equation

\[
\frac{\sigma_A(\xi', \xi_m) + \sigma_B(\xi', \xi_m)}{2} \tilde{U}(\xi', \xi_m) + \frac{\sigma_A(\xi', \xi_m) - \sigma_B(\xi', \xi_m)}{2} \times (H\tilde{U})(\xi', \xi_m) = \tilde{V}(\xi', \xi_m), \quad \xi_m \in \mathbb{R}
\]

with a parameter \(\xi' \in \mathbb{R}^{m-1}\).

The theory for such equations or equivalently Riemann boundary value problems is well-known [2, 3, 5, 7]. The index of such Riemann boundary value problem plays key role in describing construction for a solution. For our case we have only one difficulty related to a parameter \(\xi'\). We’ll give definition for the index in connection with our case.

Let’s denote

\[
\sigma(\xi', \xi_m) \equiv -\sigma_A^{-1}(\xi', \xi_m)\sigma_B(\xi', \xi_m)
\]

and suppose that \(\sigma \in C(\hat{\mathbb{R}}^m), \hat{\mathbb{R}}^m\) is a compactification of \(\mathbb{R}^m\).

**Definition 1.** Symbol \(\sigma(\xi)\) is called elliptic if \(\sigma(\xi) \neq 0, \forall \xi \in \hat{\mathbb{R}}^m\).

Fix \(\xi' \in \mathbb{R}^{m-1}\) and define

\[
\varpi(\xi') \equiv \text{Ind } \sigma = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d \arg \sigma(\xi', \xi_m).
\]

**Remark 1.** This index is an integer, and indeed it doesn’t depend on \(\xi'\) if \(m \geq 2\) (homotopy property). The case \(m = 1\) is a very specific one (see [14]). So we have \(\varpi(\xi') = \varpi\).

## 4 Solvability condition

**Definition 2.** Factorization of the elliptic symbol \(\sigma(\xi)\) is called its representation in the form

\[
\sigma(\xi) = \sigma_+(\xi) \cdot \sigma_-(\xi),
\]

where the factors \(\sigma_{\pm}\) admit an analytical continuation into complex half-planes \(\mathbb{C}_{\pm}\) and \(\sigma_{\pm} \in L_\infty(\mathbb{R})\).

Such factorization (in some more general sense) exists for all cases and can be constructed by the Hilbert transform \(H\) [2, 3, 5, 7].

Now we are ready to formulate a basic result on unique solvability of the equation (2.4).

**Theorem 1.** Let \(\sigma(\xi)\) be an elliptic symbol. Then for unique solvability of the equation (2.5) in the space \(L_2(\mathbb{R}^m)\) it is necessary and sufficient \(\varpi = 0\).

**Proof.** We see that our operator (and equation) are one-dimensional one, hence we can use one-dimensional theory (see [14]). According to the needed result the equation (2.4) has a unique solution in \(L_2(\mathbb{R}^m)\) if the condition \(\varpi = 0\) holds. \(\square\)
Of course there are another cases when the index is not zero. One-dimensional constructions for such situations are described in [14]. To adapt such constructions to multi-dimensional case we need more large scale of functional spaces than $L_2(\mathbb{R}^m_+)$. 

5 Mapping properties of difference operators

**Definition 3.** The space $H^s(\mathbb{R}^m)$, $s \in \mathbb{R}$, consists of (generalized) functions for which the following norm

$$
\|u\|_s = \left( \int_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 (1 + |\xi|)^{2s} d\xi \right)^{1/2}
$$

is a finite number.

This space is a Hilbert space, and the Schwartz space $S(\mathbb{R}^m)$ consisting of infinitely differentiable rapidly decreasing at infinity functions is dense in $H^s(\mathbb{R}^m)$ [1]. Obviously $H^0(\mathbb{R}^m) = L_2(\mathbb{R}^m)$.

**Lemma 1.** Let $\sigma_A \in L_\infty(\mathbb{R}^m)$. Then the operator $A$ is a linear bounded operator $H^s(\mathbb{R}^m) \rightarrow H^s(\mathbb{R}^m)$.

**Proof.** Indeed, for $u \in S(\mathbb{R}^m)$

$$
(FA u)(\xi) = \int_{\mathbb{R}^m} e^{ix \cdot \xi} \left( \sum_{|k|=0}^\infty a_k u(x + \alpha_k) \right) dx
$$

$$
= \sum_{|k|=0}^\infty a_k \int_{\mathbb{R}^m} e^{ix \cdot \xi} u(x + \alpha_k) dx = \sum_{|k|=0}^\infty a_k e^{i\alpha_k \cdot \xi} \tilde{u}(\xi) = \sigma_A(\xi) \cdot \tilde{u}(\xi),
$$

under the condition that sum and integral can be re-arranged, it’s possible according to our assumptions, and further

$$
\|Au\|_s^2 = \int_{\mathbb{R}^m} |\sigma_A(\xi) \cdot \tilde{u}(\xi)|^2 (1 + |\xi|)^{2s} d\xi
$$

$$
\leq \left( \text{ess sup}_{\xi \in \mathbb{R}^m} |\sigma_A(\xi)| \right)^2 \int_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 (1 + |\xi|)^{2s} d\xi = \|\sigma_A\|_{L_\infty}^2 \|u\|_s^2.
$$

□

**Remark 2.** It seems that the operator $A$ is a pseudo differential operator with the symbol $\sigma_A(\xi)$. It is so but the formula

$$
F_{\xi \rightarrow x}^{\sigma_A(\xi) \cdot \tilde{u}(\xi)}
$$

will define a very specific integral operator of convolution type with a non-regular kernel

$$
K_A(x) = \sum_{|k|=0}^\infty p_k \delta(x + \alpha_k),
$$
so that
\[ F_{\xi \to x}^{-1}(\sigma_A(\xi) \cdot \tilde{u}(\xi)) = \int_{\mathbb{R}^m} \left( \sum_{|k|=0}^{\infty} p_k \delta(x + \alpha_k - y) \right) u(y) \, dy. \]

6 Non-vanishing indices

Here we’ll apply our one-dimensional results from [14] to obtain solvability picture for a multi-dimensional case.

6.1 Positive index

Let \( \omega \in \mathbb{N} \). First we introduce a function
\[ \omega(\xi', \xi_m) = \left( \frac{\xi_m - i|\xi'|}{\xi_m + i|\xi'|} \right)^\omega, \]

which belongs to \( L_\infty(\mathbb{R}^m) \).

Evidently the functions \( z \pm i|\xi'| \) for fixed \( \xi' \in \mathbb{R}^{m-1} \) are analytical functions in complex half-planes \( \mathbb{C}_\pm \). Moreover
\[ \text{Ind} \frac{\xi_m - i|\xi'|}{\xi_m + i|\xi'|} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\arg \frac{\xi_m - i|\xi'|}{\xi_m + i|\xi'|} = 1. \]

According to the index property a function \( \omega^{-1}(\xi', \xi_m)\sigma(\xi', \xi_m) \) has a vanishing index, and it can be factorized in a sense of the definition 1:
\[ \omega^{-1}(\xi', \xi_m)\sigma(\xi', \xi_m) = \sigma_+(\xi', \xi_m)\sigma_-(\xi', \xi_m), \]

so we have \( \sigma(\xi', \xi_m) = \omega(\xi', \xi_m)\sigma_+(\xi', \xi_m)\sigma_-(\xi', \xi_m) \), where
\[ \sigma_\pm(\xi', \xi_m) = \exp(\Psi_\pm(\xi', \xi_m)), \quad \Psi_\pm(\xi', \xi_m) = \frac{1}{2\pi i} \lim_{\tau \to 0^+} \int_{-\infty}^{+\infty} \frac{\ln(\omega^{-1}\sigma(\xi, \eta_m)) d\eta_m}{\xi_m \pm i\tau - \eta_m}. \]

Further we’ll apply the Wiener–Hopf technique to the equation (2.6) and write it in a convenient form:
\[ \sigma_A(\xi)\tilde{U}_+(\xi) + \sigma_B(\xi)\tilde{U}_-(\xi) = \tilde{V}(\xi). \]

Taking into account our above calculations we have
\[ \tilde{U}_+(\xi) + \omega(\xi', \xi_m)\sigma_+(\xi', \xi_m)\sigma_-(\xi', \xi_m)\tilde{U}_-(\xi) = \sigma_A^{-1}(\xi', \xi_m)\tilde{V}(\xi), \]

or in other words
\[ \sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi) + \omega(\xi', \xi_m)\sigma_-(\xi', \xi_m)\tilde{U}_-(\xi) = \sigma_A^{-1}(\xi', \xi_m)\sigma_A^{-1}(\xi', \xi_m)\tilde{V}(\xi). \]

Let’s introduce the following notations. We denote for shortness
\[ \sigma_+^{-1}(\xi', \xi_m)\sigma_A^{-1}(\xi', \xi_m)\tilde{V}(\xi) \equiv \tilde{h}(\xi', \xi_m) \]

and define the spaces $A(\mathbb{R}^m), B(\mathbb{R}^m)$ as subspaces of functions from $L_2(\mathbb{R}^m)$ which admit an analytical continuation on the last variable $\xi_m$ under almost all fixed $\xi' \in \mathbb{R}^{m-1}$ into complex half-planes $\mathbb{C}_+, \mathbb{C}_-$ respectively so that

$$A(\mathbb{R}^m) \oplus B(\mathbb{R}^m) = L_2(\mathbb{R}^m) \quad for \ a.a. \ \xi' \in \mathbb{R}^{m-1}.$$ 

Since $\tilde{h} \in L_2(\mathbb{R}^m)$ it can be represented as a sum $\tilde{h} = h_+ + h_-$, where $h_+ \in A(\mathbb{R}^m), h_- \in B(\mathbb{R}^m)$, and $h_\pm = H \pm \hat{h}$. Thus we have

$$\sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi) + \omega(\xi', \xi_m)\sigma_- (\xi', \xi_m)\tilde{U}_-(\xi) = \tilde{h}_+ + \tilde{h}_-$$

and taking into account a form of the function $\omega(\xi', \xi_m)$ we write

$$(\xi_m + i|\xi'|)^\alpha \sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi', \xi_m) + (\xi_m - i|\xi'|)^\alpha \sigma_- (\xi', \xi_m)\tilde{U}_-(\xi', \xi_m) = (\xi_m + i|\xi'|)^\alpha \tilde{h}_+(\xi', \xi_m) + (\xi_m + i|\xi'|)^\alpha \tilde{h}_-(\xi', \xi_m)$$

and hence

$$(\xi_m + i|\xi'|)^\alpha \sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi', \xi_m) - (\xi_m - i|\xi'|)^\alpha \tilde{h}_+(\xi', \xi_m) = (\xi_m + i|\xi'|)^\alpha \tilde{h}_-(\xi', \xi_m) - (\xi_m - i|\xi'|)^\alpha \sigma_- (\xi', \xi_m)\tilde{U}_-(\xi', \xi_m). \quad (6.1)$$

The left hand side of the equation (6.1) belongs to $A(\mathbb{R}^m)$ and its analytical continuation into upper complex half-plane on the variable $\xi_m$ has a pole of order $\alpha \varepsilon$ at infinity, and the right hand side belongs to the space $B(\mathbb{R}^m)$ and has the same pole. Then taking into account the generalized Liouville theorem [2, 7] we conclude that both left hand side and right hand side is a polynomial of order no more than $\alpha \varepsilon - 1$. Since we are interested in left hand side we write

$$(\xi_m + i|\xi'|)^\alpha \sigma_+^{-1}(\xi', \xi_m)\tilde{U}_+(\xi', \xi_m) - (\xi_m - i|\xi'|)^\alpha \tilde{h}_+(\xi', \xi_m) = \sum_{k=0}^{\alpha-1} c_k(\xi')\xi_m^k,$$

where $c_k, k = 0, \cdots, \alpha - 1$ are arbitrary functions depending on a parameter $\xi'$. So we have the following formula

$$\tilde{U}_+(\xi', \xi_m) = \sigma_+(\xi', \xi_m)\tilde{h}_+(\xi', \xi_m) + (\xi_m + i|\xi'|)^{-\alpha} \sigma_+(\xi', \xi_m) \sum_{k=0}^{\alpha-1} c_k(\xi')\xi_m^k.$$

Obviously the first summand of the last formula belongs to the $L_2(\mathbb{R}^m_+)$. Since $\sigma_+ \in L_\infty(\mathbb{R}^m_+)$ then summands $(\xi_m + i|\xi'|)^{-\alpha} c_k(\xi')\xi_m^k$, should belong to the $L_2(\mathbb{R}^m_+)$, $k = 0, \cdots, \alpha - 1$. It means that

$$\int_{\mathbb{R}^m} |\xi_m + i|\xi'||^{-2\alpha} |c_k(\xi')|^2 |\xi_m|^2 d\xi < +\infty. \quad (6.2)$$

Passing to a repeated integral we first estimate a one-dimensional integral

$$\int_{-\infty}^{+\infty} |\xi_m + i|\xi'||^{-2\alpha} |\xi_m|^2 d\xi_m.$$
Corollary 1. \( L \) has the form \( \sigma \) factors

Proof. Initial point for the proof is the quality (6.1). Then we conclude that both the left hand side and the right hand side are zero because we have a zero of order \( \alpha \).

Theorem 2. Let \( \sigma(\xi) \) be an elliptic symbol, \( \alpha \) is its index of factorization with factors \( \sigma_{\pm}(\xi) \). If \( \alpha \in \mathbb{N} \) then the equation (2.5) has many solutions in the space \( L_2(\mathbb{R}^m) \), and formula for a general solution in Fourier image

\[
\tilde{U}_+(\xi', \xi_m) = \sigma_+(\xi', \xi_m) \tilde{h}_+(\xi', \xi_m) + (\xi_m + i|\xi'|)^{-\alpha} \sigma_+(\xi', \xi_m) \sum_{k=0}^{\alpha-1} c_k(\xi') \xi_m^k
\]

holds, where \( v \) is an arbitrary continuation \( v \) on \( \mathbb{R}^m \), \( c_k \in H^{s_k}(\mathbb{R}^{m-1}) \), \( s_k = -\alpha + k + 1/2, k = 0, \cdots, \alpha - 1 \), are arbitrary functions.

Corollary 1. If under assumptions of the theorem 3 \( v \equiv 0 \) then a general solution of the equation

\((Au)(x) = 0, \quad x \in \mathbb{R}^m_+\)

has the form

\[
\tilde{u}(\xi) = (\xi_m + i|\xi'|)^{-\alpha} \sigma_+(\xi', \xi_m) \sum_{k=0}^{\alpha-1} c_k(\xi') \xi_m^k.
\]

Note. If we are interested in the equation (2.4) we need to put \( \sigma(\xi) = \sigma_{\mathcal{A}}^{-1}(\xi) \) to obtain this assertion.

6.2 Negative index

Theorem 3. Let \( \sigma(\xi) \) be an elliptic symbol and \( \alpha < 0 \). The equation (2.5) has a solution \( U \in L_2(\mathbb{R}^m) \) iff the right hand side \( V \) satisfies the following conditions:

\[
\int_{-\infty}^{+\infty} \frac{\sigma_{-1}(\xi', \eta_m) \sigma_{\mathcal{A}}^{-1}(\xi', \eta_m) \tilde{V}(\xi', \eta_m) d\eta_m}{(\eta_m + i|\xi'|)^{k+1}} = 0, \quad k = 0, 1, \cdots, \lvert \alpha \rvert.
\]

Proof. Initial point for the proof is the quality (6.1). Then we conclude that both the left hand side and the right hand side are zero because we have a zero of order \( -\alpha \) at infinity. Hence

\[
(\xi_m + i|\xi'|)^{\alpha} \sigma_+(\xi', \xi_m) \tilde{U}_+(\xi', \xi_m) = (\xi_m + i|\xi'|)^{\alpha} \tilde{h}_+(\xi', \xi_m) = 0
\]

and we have

\begin{align*}
\tilde{U}_+(\xi', \xi_m) &= \sigma_+(\xi', \xi_m)\tilde{h}_+(\xi', \xi_m), \\
\tilde{U}_-(\xi', \xi_m) &= \omega^{-1}(\xi', \xi_m)\sigma_-^{-1}(\xi', \xi_m)\tilde{h}_-(\xi', \xi_m) \\
&= \left(\frac{\xi_m - i|\xi'|}{\xi_m + i|\xi'|}\right)^{-ae} \sigma_-^{-1}(\xi', \xi_m)\tilde{h}_-(\xi', \xi_m).
\end{align*}

These functions \(\tilde{U}_\pm\) belong to the spaces \(A(\mathbb{R}^m), B(\mathbb{R}^m)\) respectively but the function \(\tilde{U}_-(\xi', \xi_m)\) has a pole of order \(-ae\) in the point \(-i|\xi'|\). To exclude such possibility we’ll use the expansion of the Cauchy type integral \(\tilde{h}_-\) like [1]:

\begin{equation}
\tilde{h}_-(\xi', \xi_m) = (II_-\tilde{h})(\xi', \xi_m) = i \sum_{k=0}^{[ae]} A_+^k(\xi', \xi_m)II'\Lambda_-^{k-1}\tilde{h} + A_+^{ae}(\xi', \xi_m)II_-\Lambda_-^{ae}\tilde{h},
\end{equation}

where

\begin{equation}
A_+(\xi', \xi_m) = \xi_m + i|\xi'|, \quad (II'\tilde{h})(\xi') \equiv \int_{-\infty}^{+\infty} \tilde{h}(\xi', \xi_m)d\xi_m
\end{equation}

and notation \(\Lambda_+\tilde{h}\) denotes a product of two functions \(A_+(\xi', \xi_m)\) and \(\tilde{h}(\xi', \xi_m)\).

Thus the pole in the point \(-i|\xi'|\) will disappear if the following conditions hold:

\begin{equation}
\int_{-\infty}^{+\infty} \frac{\tilde{h}(\xi', \eta_m)d\eta_m}{(\eta_m + i|\xi'|)^{k+1}} = 0 \quad \text{for a.a. } \xi' \in \mathbb{R}^{m-1}, \quad k = 0, 1, \ldots, [ae].
\end{equation}

Taking into account that

\begin{equation}
\tilde{h}(\xi', \xi_m) = \sigma_+^{-1}(\xi', \xi_m)\sigma_-^{-1}(\xi', \xi_m)\tilde{V}(\xi)
\end{equation}

and substituting it in obtained conditions we have

\begin{equation}
\int_{-\infty}^{+\infty} \frac{\sigma_+^{-1}(\xi', \eta_m)\sigma_-^{-1}(\xi', \eta_m)\tilde{V}(\xi', \eta_m)d\eta_m}{(\eta_m + i|\xi'|)^{k+1}} = 0.
\end{equation}

**Remark 3.** If we’ll enlarge the space scale and consider the Sobolev–Slobodetskii spaces \(H^s\) then we’ll can omit these conditions. But since this problem will be over-determined we need some additional unknowns. These are so-called co-boundary operators or potential like operators [1]. We’ll study these situations in a separate paper.

7 Variable coefficients

Here we consider a difference operator with variable coefficients in the space \(L_2(\mathbb{R}^m)\):

\begin{equation}
(Du)(x) = \sum_{|k|=0}^{+\infty} p_k(x)u(x + \gamma_k), \quad x \in \mathbb{R}^m, \quad \gamma_k = (\gamma_{k_1}, \ldots, \gamma_{k_m}) \in \mathbb{R}^m. \quad (7.1)
\end{equation}
Definition 4. The function
\[ \sigma_D(x, \xi) = \sum_{|k|=0}^{+\infty} p_k(x) e^{i\gamma_k \cdot \xi} \]
is called a symbol of the operator \( D \).

We'll suppose in this section that all functions \( p_k(x), |k| = 0, 1, \ldots \) are continuous in \( \mathbb{R}^m \), denote
\[ \|p_k\|_{C(\mathbb{R}^m)} = p_k, \]
assuming
\[ \sum_{|k|=0}^{+\infty} p_k = A < +\infty. \]

Lemma 2. The operator \( D \) is a linear bounded operator \( L_p(\mathbb{R}^m) \longrightarrow L_p(\mathbb{R}^m) \), \( 1 \leq p \leq +\infty \).

Proof. We have
\[ |(Du)(x)| \leq \sum_{|k|=0}^{+\infty} |p_k(x)| |u(x + \gamma_k)| \leq \sum_{k=0}^{+\infty} p_k |u(x + \gamma_k)| \]
and then
\[ \|Du\|_1 = \int_{\mathbb{R}^m} |(Du)(x)| dx \leq \sum_{k=0}^{+\infty} p_k \int_{\mathbb{R}^m} |u(x + \gamma_k)| \leq A \|u\|_1, \]
i.e. \( D : L_1(\mathbb{R}^m) \longrightarrow L_1(\mathbb{R}^m) \) is a linear bounded operator.

In the same way we conclude that \( D : L_\infty(\mathbb{R}^m) \longrightarrow L_\infty(\mathbb{R}^m) \). Further we can apply simplest interpolation theorems [10, 18] and conclude that the operator \( D : L_p(\mathbb{R}^m) \longrightarrow L_p(\mathbb{R}^m) \) is a linear bounded operator for all \( 1 \leq p \leq +\infty \).

7.1 Local principle

For every operator \( D \) we define an operator family \( \{D_{x_0}\}_{x_0 \in \mathbb{R}^m} \) where
\[ D_{x_0} : u(x) \longrightarrow \sum_{|k|=0}^{+\infty} p_k(x_0) u(x + \gamma_k), \quad x \in \mathbb{R}^m \]
and the function
\[ \sigma(x_0, \xi) = \sum_{|k|=0}^{+\infty} p_k(x_0) e^{i\gamma_k \cdot \xi} \]
will be its symbol (see (2.2)) because the operator \( D_{x_0} \) is an operator with constant coefficients. Following Simonenko [9] we call the operator \( D_{x_0} \) a local representative of the operator \( D \) in the point \( x_0 \).
Let \( \mathcal{L}(D) \) be a space of linear bounded operators of type (7.1) with operator norm \( \|D\|_{L^p(\mathbb{R}^m) \to L^p(\mathbb{R}^m)}, 1 \leq p \leq +\infty \). Thus we can introduce the operator function \( D(x) \equiv D_x \) for all \( x \in \mathbb{R}^m \).

We remind that an operator \( D \) has a Fredholm property if

\[
\text{Ind } A \equiv \dim \text{Ker } A - \dim \text{Coker } A
\]

is a finite number. A Fredholm property is stable under compact and small perturbations, and \( \text{Ind } A \) is a homotopic invariant of the operator \( A \) [5].

**Lemma 3.** Operator \( D \) has a Fredholm property in the space \( L^p(\mathbb{R}^m) \) iff the family \( \{D_{x_0}\}_{x_0 \in \mathbb{R}^m} \) consists of invertible operators.

**Proof.** Let \( D \) be an algebra of difference operators of type (7.1), and \( \Psi \) be an algebra of pseudo differential operators with symbols \( p(x, \xi) \in C(\mathbb{R}^m \times \mathbb{R}^m) \). Obviously \( \mathcal{D} \subset \Psi \). Further let \( P(x) \) be a corresponding operator function for a pseudo differential operator \( P \in \Psi \). We say that operator function is invertible if exists an operator function \( P^{-1}(x) \) such that

\[
P(x)P^{-1}(x) = I, \quad \forall x \in \mathbb{R}^m.
\]

Now our main goal is to show that given operator function \( D(x) \) one can construct a pseudo differential operator \( P \in \Psi \) such that \( DP \) and \( PD \) can be represented in the form \( I + T \), where \( T \) is a compact operator. It will imply that the operator \( D \) has a two-sided regularizer, and thus will have a Fredholm property [5].

In the algebra \( \Psi \) we’ll extract a special type of operators so called operators of a local type [9]. Without going into details let’s say that pseudo differential operator from \( P \in \Psi \) will be an operator of a local type if its kernel

\[
K_P(x, y) = F_{\xi \to y}^{-1} p(x, \xi)
\]

is at least a continual function of variables \( (x, y) \) and generates a linear bounded operator of the type

\[
u(x) \mapsto \int_{\mathbb{R}^m} K_P(x, x - y)u(y)dy.
\]

Further we’ll correct a symbol \( p(x, \xi) \in C(\mathbb{R}^m \times \mathbb{R}^m) \) by small perturbations so that the corrected pseudo differential operator \( P_\varepsilon \) with a symbol \( p_\varepsilon(x, \xi) \) will generate the continual kernel \( K_{P_\varepsilon}(x, \xi) \). We need two steps.

First if \( p(\infty) = c \) then we consider

\[
p_1(x, \xi) = p(x, \xi) - c,
\]

so that \( p_1(\infty) = 0 \), and \( p(x, \xi) = p_1(x, \xi) + c \). It means that the operator with the symbol \( p(x, \xi) \) is represented in the form \( P_1 + cf \).

Second one can approximate the symbol \( p_1(x, \xi) \) by symbols \( p_{1, \varepsilon}(x, \xi) \in S(\mathbb{R}^m \times \mathbb{R}^m) \) (Schwartz class) with smooth kernel \( K_{P_{1, \varepsilon}}(x, y) \). Such operator

\[
(P_\varepsilon u)(x) = cu(x) + \int_{\mathbb{R}^m} K_{P_{1, \varepsilon}}(x, x - y)u(y)dy
\]
is an operator of a local type, and
\[ \| P - P_\varepsilon \|_{L_p(\mathbb{R}^m) \to L_p(\mathbb{R}^m)} \leq \varepsilon. \]

It’s well known [5, 9] that the family \( \{ P_\varepsilon(x) \} \) of operators of a local type reconstructs the operator \( P_\varepsilon \) up to compact summand. Moreover now we can assert the following. If the operator function \( P_\varepsilon(x) \) is invertible then the operator \( P_\varepsilon^{-1} \) with the symbol \( p_\varepsilon^{-1}(x, \xi) \) will be two-sided regularizer for the operator \( P_\varepsilon \) and consequently for \( P \). □

**Corollary 2.** Operator \( D : L_p(\mathbb{R}^m) \to L_p(\mathbb{R}^m) \) has a Fredholm property iff
\[ \sigma_D(x, \xi) \neq 0, \quad \forall x, \xi \in \mathbb{R}^m. \]

**Lemma 4.** Index of a Fredholm operator \( D : L_p(\mathbb{R}^m) \to L_p(\mathbb{R}^m) \) is equal to 0.

**Proof.** According to convexity of \( \mathbb{R}^m \) one can easily construct a homotopy
\[ D_t, x_0 \equiv D(1-t)x_0 + tx, \quad t \in [0, 1], \]
where \( x_0 \in \mathbb{R}^m \) is an arbitrary fixed point. Since all intermediate operators are invertible then an initial operator \( D = D_{1, x_0} \) has vanishing index. □

### 7.2 Half-space case

This section is connected to operators in \( D : L_2(\mathbb{R}^m_+) \to L_2(\mathbb{R}^m_+) \). These construction are more complicated but a general idea is the same. The main result for this case is the following.

**Theorem 4.** Operator \( D : L_2(\mathbb{R}^m_+) \to L_2(\mathbb{R}^m_+) \) has a Fredholm property iff the following conditions
\begin{align*}
1) \quad &\sigma_D(x, \xi) \neq 0, \quad \forall x \in \mathbb{R}^m_+, \quad \xi \in \mathbb{R}^m, & 2) \quad \int_{-\infty}^{+\infty} d\arg \sigma_D(\cdot, \cdot, \xi_m) = 0.
\end{align*}
hold.

**Sketch of proof.** In this case also a local principle plays key role. But main lemma describing the local principle will be distinct. Namely for our difference operator \( D \) the operator function will consist of two parts. The first part \( D_1(x) \) will be defined on \( \mathbb{R}^m_+ \) and it generated by operators of type (7.2); more precisely it consists of pseudo differential operators with symbols \( \sigma_D(x, \xi) \):
\[ u(x) \mapsto F_{\xi \to x}^{-1} \sigma_D(x, \xi) \tilde{u}(\xi). \]

The second part \( D_2(x) \) consists of operators of the following type
\[ u(x) \mapsto \int_{\mathbb{R}^m_+} K_D(x, x - y) u(y) dy, \]
or in more general form and in Fourier image
\[ \tilde{u}(\xi) \mapsto \sigma_A(x, \xi)(\Pi_+ \tilde{u})(x, \xi) + \sigma_B(x, \xi)(\Pi_- \tilde{u})(x, \xi). \]

(We’ll remind here that for the operator \( D : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m) \) one should put in the last formula \( \sigma_A(x, \xi) = \sigma_D(x, \xi), \sigma_B(x, \xi) = 1 \). The operator function \( D_2(x) \) will be defined for the points \( x \in \mathbb{R}^{m-1} \), i.e for boundary points of \( \mathbb{R}^m_+ \). Further we should prove that operator \( D \) has a Fredholm property iff both parts \( D_1(x) \) and \( D_2(x) \) are invertible for all admissible \( x \). If so then the condition 1 in the theorem is an invertibility condition for the first operator family, and the condition 2 is the same for the second part. \( \square \)

Conclusions

There are many other interesting cases for studying solvability for difference equations in special canonical domain like multidimensional cones. This will be the object of another paper.

Acknowledgements

We would like to thank referees for their helpful remarks.

References


