A Joint Elliott Type Theorem for Twists of $L$-Functions of Elliptic Curves

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**Abstract.** We consider a collection of $L$-functions of elliptic curves twisted by a Dirichlet character modulo $q$ ($q$ is a prime number), and prove for this collection a joint limit theorem for weakly convergent probability measures in the space of analytic functions as $q \to \infty$. The limit measure is given explicitly.

**Keywords:** Dirichlet character, elliptic curve, $L$-function of elliptic curve, probability measure, weak convergence.

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1 Introduction

Let $E$ be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

with discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$. For each prime number $p$, denote by $E_p$ the reduction modulo $p$ of the curve $E$ which is a curve over the finite field $\mathbb{F}_p$, and define the integer $\lambda(p)$ by the equality

$$|E(\mathbb{F}_p)| = p + 1 - \lambda(p),$$

where $|E(\mathbb{F}_p)|$ is the number of points of curve $E_p$. The $L$-function $L_E(s)$, $s = \sigma + it$, of the curve $E$ is defined by the Euler product

$$L_E(s) = \prod_{p | \Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1},$$

which, in virtue of the estimate

$$|\lambda(p)| \leq 2\sqrt{p}$$
is absolutely convergent for \( \sigma > 3/2 \). Moreover, the function \( L_E(s) \) is analytically continued to an entire function, see, for example, [12].

Now let \( \chi \) be a Dirichlet character modulo \( q \). Then the twist \( L_E(s, \chi) \) of the function \( L_E(s) \) is defined, for \( \sigma > 3/2 \), by the Euler product

\[
L_E(s, \chi) = \prod_{p|\Delta} \left( 1 - \frac{\lambda(p)\chi(p)}{p^s} \right)^{-1} \prod_{p\not|\Delta} \left( 1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}} \right)^{-1}
\]

and can be expanded in the Dirichlet series

\[
\sum_{m=1}^{\infty} \frac{\lambda(m)\chi(m)}{m^s}.
\]

In the sequel, we assume that \( q \) is a prime number. Then it was observed in [12] that the function \( L_E(s, \chi) \), as \( L_E(s) \), is also entire one.

Limit theorems for \( L_E(s, \chi) \) with increasing \( q \) were began to study in [6], [7], [8], [9], however, only in the half-plane of absolute convergence \( \sigma > 3/2 \). In [12], a limit theorem in the space of analytic functions \( H(D) \), \( D = \{ s \in \mathbb{C} : \sigma > 1 \} \), for the function \( L_E(s, \chi) \) has been obtained. For its statement, we need some notation and definitions. For \( Q \geq 2 \), let

\[
M_Q = \sum_{q \leq Q} \sum_{\chi \neq \chi_0} 1,
\]

where, as usual, \( \chi_0 \) denotes the principal character modulo \( q \). It is well known that

\[
M_Q = \frac{Q^2}{2\log Q} + \mathcal{O}\left( \frac{Q^2}{\log^2 Q} \right).
\]

Denote by \( \gamma \) the unite circle \( \{ s \in \mathbb{C} : |s| = 1 \} \), and define \( \Omega = \prod_p \gamma_p \), where \( \gamma_p = \gamma \) for all primes \( p \). The infinite-dimensional torus \( \Omega \) with the product topology and operation of pointwise multiplication is a compact topological Abelian group. Therefore, on \( (\Omega, \mathcal{B}(\Omega)) \), where \( \mathcal{B}(X) \) is the Borel \( \sigma \)-field of the space \( X \), the probability Haar measure \( m_H \) exists, and this gives the probability space \( (\Omega, \mathcal{B}(\Omega), m_H) \). Denote by \( \omega(p) \) the projection of an element \( \omega \in \Omega \) to the coordinate space \( \gamma_p \), and, on \( (\Omega, \mathcal{B}(\Omega), m_H) \), define the \( H(D) \)-valued random element \( L_E(s, \omega) \) by the formula

\[
L_E(s, \omega) = \prod_{p|\Delta} \left( 1 - \frac{\lambda(p)\omega(p)}{p^s} \right)^{-1} \prod_{p\not|\Delta} \left( 1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}} \right)^{-1}.
\]

Let \( P_{L_E} \) be the distribution of \( L_E(s, \omega) \), i. e.,

\[
P_{L_E}(A) = m_H(\omega \in \Omega : L_E(s, \omega) \in A), \quad A \in \mathcal{B}(H(D)).
\]

Then the main result of [12] is the following limit theorem.

**Theorem 1.** Suppose that \( Q \to \infty \). Then
\[
\frac{1}{M_Q} \# \{ \chi(\mod q), q \leq Q, \chi \neq \chi_0 : L_E(s, \chi) \in A \}, \quad A \in \mathcal{B}(H(D))
\]
converges weakly to \( P_{L_E} \).

Here \( \#A \) stands for the cardinality of the set \( A \).

In [8], a joint limit theorem of type of Theorem 1 has been obtained for a collection of moduli of twists of \( L \)-functions of elliptic curves, however, in the region \( \sigma > \frac{3}{2} \), only. We note that P. D. T. A. Elliott was the first who began to study limit theorems with increasing modulus for Dirichlet \( L \)-functions [4], [5].

The aim of this paper is a multidimensional analogue of Theorem 1. For \( j = 1, \ldots, r \), let \( E_j \) be an elliptic curve over the field of rational numbers given by the equation
\[
y^2 = x^3 + a_j x + b_j, \quad a_j, b_j \in \mathbb{Z}
\]
with discriminant \( \Delta_j = -16(4a_j^3 + 27b_j^2) \neq 0 \). Consider the corresponding \( L \)-function
\[
L_{E_j}(s) = \prod_{p | \Delta_j} \left( 1 - \frac{\lambda_j(p)}{p^s} \right)^{-1} \prod_{p \nmid \Delta_j} \left( 1 - \frac{\lambda_j(p)}{p^s} + \frac{1}{p^{2s-1}} \right)^{-1}.
\]

Suppose that \( N_j \) is the conductor of the curve \( E_j \). Then, by the Weil-Shimura-Taniyama conjecture proved in [3], see also Theorem 14.6 of [10], the function \( L_{E_j}(s) \) coincides with \( L \)-function of a new cusp form of weight 2 and level \( N_j \). This shows that \( L_{E_j}(s) \) is an entire function.

By Theorem 14.20 of [10], the twist \( L_{E_j}(s, \chi) \) of \( L_{E_j}(s) \) with a character \( \chi \) modulo \( q \) is again a new cusp form of weight 2 and level \( N_j q^2 \). Therefore, the function \( L_{E_j}(s, \chi) \) is also an entire function.

Let, for brevity, \( E = (E_1, \ldots, E_r) \), \( L_E(s, \chi) = (L_{E_1}(s, \chi), \ldots, L_{E_r}(s, \chi)) \) and \( A_Q = \{ \chi(\mod q) : q \leq Q, \chi \neq \chi_0 \} \). Moreover, define
\[
L_E(s, \omega) = (L_{E_1}(s, \omega), \ldots, L_{E_r}(s, \omega)),
\]
where, for \( j = 1, \ldots, r \),
\[
L_{E_j}(s, \omega) = \prod_{p | \Delta_j} \left( 1 - \frac{\lambda_j(p)\omega(p)}{p^s} \right)^{-1} \prod_{p \nmid \Delta_j} \left( 1 - \frac{\lambda_j(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}} \right)^{-1}.
\]

Denote by \( P_{L_E} \) the distribution of the \( H^r(D) \)-valued random element \( L_E(s, \omega) \), i.e.,
\[
P_{L_E}(A) = m_H \left\{ \omega \in \Omega : L_E(s, \omega) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).
\]
Then we have the following statement.

**Theorem 2.** Suppose that \( Q \to \infty \). Then
\[
P_{Q,E}(A) \overset{\text{def}}{=} \frac{1}{M_Q} \# \{ \chi \in A_Q : L_E(s, \chi) \in A \}, \quad A \in \mathcal{B}(H^r(D))
\]
converges weakly to \( P_{L_E} \).
2 Auxiliary results

We start with a joint limit theorem which is a generalization of Lemma 1 from [12]. We extend the function $\omega(p)$ to the set $\mathbb{N}$ by the formula

$$\omega(m) = \sum_{p^j | m, p^{j+1} \nmid m} \omega^j(p), \quad m \in \mathbb{N}.$$ 

Lemma 1. For $j = 1, \ldots, r$, let \{a_{mj} : m \in \mathbb{N}\} be a sequence of complex numbers such that

$$\sum_{m \leq n} |a_{mj}|^2 = \mathcal{O}(n^{2\alpha}), \quad \alpha > 0$$

as $n \to \infty$, and, for $\omega \in \Omega$ and $\sigma > \alpha + \frac{1}{2}$, let $X_j(s, \omega) = \sum_{m=1}^{\infty} \frac{a_{mj}\omega(m)}{m^s}$.

Suppose that \{A_m : m \in \mathbb{N}\} is a sequence of finite subsets of the torus $\Omega$ such that, for each $\omega \in \bigcup_{m=1}^{\infty} A_m$, $X_j(s, \omega)$ has an analytic continuation to the half plane $D_\alpha = \{s \in \mathbb{C} : \sigma > \alpha\}$ satisfying the following conditions:

1. \[ \sum_{\omega \in A_m} |X_j(\sigma + it, \omega)|^2 = \mathcal{O}(|t|^A), \quad A > 0 \]
   uniformly for $m \in \mathbb{N}$ and $\sigma$ in compact subsets of the interval $(\alpha, \infty)$;

2. \[ \sum_{\omega \in A_m} |X_j(\sigma, \omega)|^2 = \mathcal{O}(\#A_m) \]
   as $m \to \infty$, uniformly for $s$ on compact subsets of $D_\alpha$;

Moreover, suppose that

$$\frac{\#\{A \cap A_m\}}{\#A_m}, \quad A \in \mathcal{B}(\Omega)$$

converges weakly to the Haar measure $m_H$. Then

$$\frac{1}{\#A_m} \#\{\omega \in A_m : (X_1(s, \omega), \ldots, X_r(s, \omega)) \in A\},$$

where $A \in H^r(D_\alpha)$, converges weakly to the distribution of the random element $(X_1(s, \omega), \ldots, X_r(s, \omega))$ as $m \to \infty$.

Proof. A way of the proof is completely analogical to that in one-dimensional case presented in [2], Proposition 4.4.1. In our case, the metric in $H^r(D_\alpha)$ inducing its topology of uniform convergence on compacta is applied, and the joint case is reduced to the one-dimensional case. $\square$

The next lemma is devoted to checking the hypotheses of Lemma 1, and contains an approximate functional equation of $L$-functions of cusp forms of weight 2 and level $N$. Let $F(z)$ be a new form of weight 2 and level $N$ with
Fourier coefficients $c(m)$. Moreover, let $\Gamma(s)$, as usual, denote the gamma-function, and let $\Gamma(s, z)$ be the incomplete gamma-function,

$$\Gamma(s, z) = \int_z^{\infty} e^{-t} t^{s-1} dt, \quad \sigma > 0, \quad z \in \mathbb{R}.$$  

**Lemma 2.** [1]. Suppose that $L(s, F)$, $s = \sigma + it$, is the $L$-function associated to the form $F$, $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, $M > \frac{1\sqrt{N}}{4}$, $r = e^{i(\frac{\pi}{2} - \delta(t))}$ with $0 < \delta(t) \leq \frac{\pi}{2}$. Then

$$L(s, F) = \frac{1}{\Gamma(s)} \sum_{m \leq M} \frac{c(m)}{m^s} \Gamma\left(s, \frac{2\pi mr}{\sqrt{N}}\right) - \frac{\mu N^{1-s}(2\pi)^{2(s-1)}}{\Gamma(s)} \sum_{m \leq M} \frac{c(m)}{m^{2-s}} \Gamma\left(2 - s, \frac{2\pi m}{\sqrt{N}r}\right) + \frac{(2\pi)^s}{\Gamma(s)} R,$$

where

$$|R| < e^{-\frac{\pi t}{2}} e^{\delta(t)\left(t - \frac{4\log \pi}{N}\right)} N^{1-s} \sqrt{M} \delta^{-1}(t) \times \left(1 + \frac{\log M + \sigma + 1}{2t \delta(t)} + \frac{(\sigma - 1)(\log M + 2)}{4(\delta(t))^2}\right).$$

Let $G$ be a compact Abelian group. Then, on $(G, \mathcal{B}(G))$, the probability Haar measure $\mu$ can be defined. We recall that a sequence $\{x_m : m \in \mathbb{N}\} \subset G$ is said to be uniformly distributed if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} f(x_m) = \int_G f d\mu$$

for any real bounded Borel measurable function $f$.

The next lemma is a criterion of uniform distribution for sequences in $G$.

**Lemma 3.** The sequence $\{x_m : m \in \mathbb{N}\} \subset G$ is uniformly distributed in $G$ if and only if, for any nontrivial character $\chi_G$, the equality

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \chi_G(x_m) = 0$$

holds.

The proof of the lemma can be found in [11], Chapter 4, Corollary 1.2.

### 3 Proof of Theorem 2

In the notation of Lemma 1, we have that $a_{mj} = \lambda_j(m) \chi(m)$. Since $\lambda_j(m)$ coincides with Fourier coefficients of a new cusp form, we have, by the estimate (14.53) from [10], that

$$\sum_{m \leq n} |a_{mj}|^2 = O(n^2).$$
Therefore, $\alpha = 1$ in Lemma 1.

Denote by $\mathbb{P}$ the set of all prime numbers. Let $\chi$ be a Dirichlet character modulo $q \in \mathbb{P}$, and

$$\hat{\chi}(p) = \begin{cases} \chi(p), & \text{if } p \in \mathbb{P} \smallsetminus \{q\}, \\ 1, & \text{if } p = q. \end{cases}$$

Then $\{\hat{\chi}(p) : p \in \mathbb{P}\}$ is an element of the torus $\Omega$. Putting

$$l_{E_j, q}(s) = \begin{cases} 1 - \frac{\lambda_j(q)}{q^s}, & \text{if } q \mid \Delta, \\ 1 - \frac{\lambda_j(q)}{q^s} + \frac{1}{q^{s-1}}, & \text{if } q \not\mid \Delta, \end{cases}$$

we have that

$$L_{E_j}(s, \chi) = l_{E_j, q}(s) L_{E_j}(s, \hat{\chi}). \quad (3.1)$$

Denote by $p_m$ the $m$-th prime number, and define $A_m = \{\chi(\text{mod} p_m) : \chi \neq \chi_0\}$. Obviously, $\# A_m = p_m - 2$. Define one more set $\hat{A}_m = \{\hat{\chi} : \chi \in A_m\}$. On a certain probability space $(\hat{\Omega}, A, P)$, define the $H^r(D)$-valued random elements $X_m(s)$ and $\hat{X}_m(s)$ by the formulae

$$P\left(X_m(s) = L_E(s, \chi)\right) = \frac{1}{p_m - 2}, \quad \chi \in A_m,$$

$$P\left(\hat{X}_m(s) = L_E(s, \hat{\chi})\right) = \frac{1}{p_m - 2}, \quad \hat{\chi} \in \hat{A}_m.$$

By the definition of $l_{E_j, p_m}(s)$, we see that

$$l_{E_j, p_m}(s) \to 1 \quad (3.2)$$

in the space $H(D)$ as $m \to \infty$. Hence, $X_m(s)$ converges in distribution to $\hat{X}_m$ as $m \to \infty$. Therefore, for the proof that

$$Q_q(A) \overset{\text{def}}{=} \frac{1}{q - 2} \# \{\chi(\text{mod} q), \chi \neq \chi_0 : L(s, \chi) \in A\}, \quad A \in \mathcal{B}(H^r(D))$$

converges weakly to $P_L$ as $q \to \infty$ it suffices to obtain that the random element $\hat{X}_m$ converges in distribution to $P_{L_E}$. Thus, the sequence $\{\hat{A}_m : m \in \mathbb{N}\}$ corresponds the sequence $\{A_m : m \in \mathbb{N}\}$ in Lemma 1, and $X(s, \omega) = L_E(s, \hat{\chi})$. Moreover, in virtue of (3.1) and (3.2), $L_E(s, \hat{\chi})$ can be replaced by $L_E(s, \chi)$.

It remains to check other hypotheses of Lemma 1. Let $K$ be a compact subset of $(1, \infty)$. We continue with estimate for

$$D_q(\sigma, t) \overset{\text{def}}{=} \frac{1}{q - 2} \sum_{\chi(\text{mod} q)} |L_{E_j}(\sigma + it, \chi)|^2, \quad t > 0,$$

when $q$ runs prime numbers. For this, we apply Lemma 2 with $M = ct \sqrt{N_j} \times q \log^2 q$, $c > 0$, and $\delta = t^{-1}$. We have

$$D_q(\sigma, t) \ll_K D_{1,q}(\sigma, t) + D_{2,q}(\sigma, t) + D_{3,q}(\sigma, t), \quad (3.3)$$
where
\[
D_{1,q}(\sigma,t) = \frac{1}{q} \sum_{\chi(\text{mod}q)} \frac{1}{|\Gamma(s)|^2} \left| \sum_{m \leq M} \frac{\lambda(m)\chi(m)}{m^{\sigma+it}} \Gamma\left(\sigma + it, \frac{2\pi m r}{\sqrt{Nj\sigma}}\right) \right|^2
\]
\[\ll \frac{1}{|\Gamma(s)|^2} \sum_{m \leq M} \frac{|\lambda(m)|^2}{m^{2\sigma}} \left| \Gamma\left(\sigma + it, \frac{2\pi m r}{\sqrt{Nj\sigma}}\right) \right|^2 \frac{1}{q} \sum_{\chi(\text{mod}q)} |\chi(m)|^2
\]
\[+ \frac{1}{|\Gamma(s)|^2} \sum_{m \leq M} \sum_{n \leq M, n \neq m} \frac{|\lambda(m)||\lambda(n)|}{m^{\sigma}n^{\sigma}} \left| \Gamma\left(\sigma + it, \frac{2\pi m r}{\sqrt{Nj\sigma}}\right) \right| \times \left| \Gamma\left(\sigma + it, \frac{2\pi n r}{\sqrt{Nj\sigma}}\right) \right| \frac{1}{q} \sum_{\chi(\text{mod}q)} |\chi(m)\chi(n)|, \tag{3.4}\]
\[
D_{2,q}(\sigma,t) = \frac{1}{q} \sum_{\chi(\text{mod}q)} \frac{N_j^{2-2\sigma}q^{4-4\sigma}}{|\Gamma(s)|^2} \left| \sum_{m \leq M} \frac{\lambda(m)\chi(m)}{m^{2-\sigma-it}} \Gamma\left(2 - \sigma - it, \frac{2\pi m}{\sqrt{Njqr}}\right) \right|^2
\]
\[= \frac{N_j^{2-2\sigma}q^{4-4\sigma}}{|\Gamma(s)|^2} \left| \sum_{m \leq M} \frac{|\lambda(m)|^2}{m^{4-2\sigma}} \Gamma\left(2 - \sigma - it, \frac{2\pi m}{\sqrt{Njqr}}\right) \right|^2 \frac{1}{q} \sum_{\chi(\text{mod}q)} |\chi(m)|^2
\]
\[+ \frac{N_j^{2-2\sigma}q^{4-4\sigma}}{|\Gamma(s)|^2} \sum_{m \leq M} \sum_{n \leq M, n \neq m} \frac{|\lambda(m)||\lambda(n)|}{m^{2-\sigma}n^{2-\sigma}} \left| \Gamma\left(2 - \sigma - it, \frac{2\pi m}{\sqrt{Njqr}}\right) \right| \times \left| \Gamma\left(2 - \sigma - it, \frac{2\pi n r}{\sqrt{Njqr}}\right) \right| \frac{1}{q} \sum_{\chi(\text{mod}q)} |\chi(m)\chi(n)|, \tag{3.5}\]
\[
D_{3,q}(\sigma,t) = \frac{1}{|\Gamma(s)|^2} R^2. \tag{3.6}\]

Let \(d(m) = \sum_{k | m} 1\) be the divisor function. Using the well-known bounds
\[
|\lambda(m)| \leq \sqrt{md(m)}, \quad \sum_{m \leq x} d^2(m) \ll x \log^4 x
\]
as well as [1]
\[
|\Gamma(\sigma + it, Ar)| \ll A^\nu e^{-\left(\frac{x}{2} - \delta(t)\right)t},
\]
and the properties of the gamma-function, we find that the first term in the right-hand side of (3.4) is estimated as
\[
\ll K q^{2-2\sigma}(\log q)^{c_1} t^{A_1} \ll K t^{A_1}, \quad c_1 > 0, \quad A_1 > 0 \tag{3.7}\]
uniformly in \(\sigma \in K\) and \(q\). Moreover, in view of the equalities
\[
\sum_{\chi(\text{mod}q)} \chi(m)\overline{\chi(n)} = \begin{cases} q - 1, & \text{if } m \equiv n(\text{mod}q), \\ 0, & \text{if } m \not\equiv n(\text{mod}q) \end{cases} \tag{3.8}\]
and the bound $d(m) \ll m^\varepsilon$, $\varepsilon > 0$, we obtain that the second term in the right-hand side of (3.4) has a bound

$$
\ll \sum_{m \leq M} \sum_{n \equiv m \pmod{q}} \left| \frac{\lambda(n)}{n^\sigma} \right| \left( \frac{m}{q} \right)^\sigma \left( \frac{n}{q} \right)^\sigma
$$

$$
\ll \frac{1}{q^{2\sigma}} \sum_{k \leq M/q} \sum_{m \leq M} \sqrt{md(m)} \sqrt{m + kq} \, d(m + kq)
$$

$$
\ll q^{2-2\sigma+\varepsilon} t^{A_2} \ll_K t^{A_2}, \quad A_2 > 0
$$

uniformly in $\sigma \in K$ and $q$. This together with (3.7) shows that

$$
D_{1,q}(\sigma, t) \ll_K t^{A_3}, \quad A_3 > 0 \tag{3.9}
$$

uniformly in $\sigma \in K$ and $q$.

In a similar way, we obtain that

$$
D_{2,q}(\sigma, t) \ll_K t^{A_4}, \quad A_4 > 0 \tag{3.10}
$$

uniformly in $\sigma \in K$ and $q$. The definition of $R$ and the choice of $M$ and $\delta(t)$ imply the estimate

$$
D_{3,q}(\sigma, t) \ll_K e^{8c\log^2 q} q^{1-\sigma} \sqrt{q} (\log q)^{c_3} t^{A_5} \ll_K t^{A_5}, \quad A_5 > 0
$$

uniformly in $\sigma \in K$ and $q$. From this, (3.9), (3.10), and (3.3) we have that

$$
D_q(\sigma, t) \ll_K t^{A}, \quad A > 0
$$

uniformly in $\sigma \in K$ and $q$. Obviously, then

$$
\frac{1}{q-2} \sum_{\chi \not\equiv \chi_0 \pmod{q}} \left| L_{E_j}(\sigma + it, \chi) \right|^2 \ll_K |t|^A
$$

uniformly in $\sigma \in K$ and $q$.

In a similar manner, we obtain that

$$
\sum_{\chi \not\equiv \chi_0 \pmod{q}} \left| L_{E_j}(\sigma, \chi) \right|^2 = O(q)
$$

as $q \to \infty$, uniformly for $s$ on compact subsets of the half-plane $D$.

Next we will consider the sequence $\{A_m : m \in \mathbb{N}\}$, and will prove that it is uniformly distributed. Let $\chi_\Omega$ be a character of the group $\Omega$. Then it is well known that

$$
\chi_\Omega(\omega) = \prod_p \omega^{k_p}(p), \quad \omega \in \Omega;
$$
where only a finite number of integers \( k_p \) are distinct from zero. Hence,
\[
\chi_\Omega(\omega) = \omega(m_1) \overline{\omega(m_2)}
\]
with \((m_1, m_2) = 1\). Then we have
\[
\frac{1}{\# \hat{A}_m} \sum_{\omega \in \hat{A}_m} \chi_\Omega(\omega) = \frac{1}{p_m - 2} \sum_{\omega \in \hat{A}_m} \omega(m_1) \overline{\omega(m_2)} = \frac{1}{p_m - 2} \sum_{\chi \equiv \chi_0 \pmod{p_m}, \chi \neq \chi_0} \hat{\chi}(m_1) \overline{\hat{\chi}(m_2)}. \tag{3.11}
\]
The numbers \( m_1, m_2 \in \mathbb{N} \) are fixed. Therefore, for sufficiently large \( m \),
\[
\binom{p_m}{m_1, m_2} \nmid m_1, \ binom{p_m}{m_1, m_2} \nmid m_2, \ binom{p_m}{m_1, m_2} \nmid (m_1 - m_2).
\]
Thus, taking into account (3.11) and (3.8), we find that
\[
\frac{1}{\# \hat{A}_m} \sum_{\omega \in \hat{A}_m} \chi_\Omega(\omega) = -\frac{1}{p_m - 2} + \frac{1}{p_m - 2} \sum_{\chi \equiv \chi_0 \pmod{p_m}} \chi(m_1) \overline{\chi(m_2)}
\]
\[
= -1/(p_m - 2) \to 0
\]
as \( m \to \infty \). Therefore, by Lemma 3, the sequence \( \{\hat{A}_m : m \in \mathbb{N}\} \) is uniformly distributed. Thus, all hypotheses of Lemma 1 are fulfilled, and we have that
\[
\frac{1}{q - 2} \# \chi_{\mod {q}}, \chi \neq \chi_0 : L(s, \hat{\chi}) \in A, \quad A \in B(H^r(D))
\]
converges weakly to \( P_{L_E} \) as \( q \to \infty \), and this is true for \( Q_q \) as well.

From the weak convergence of \( Q_q \) to \( P_{L_E} \) as \( q \to \infty \), it follows that of \( P_{Q,E} \) as \( Q \to \infty \), see [12].

References


