An Extension of the Product Integration Method to $L^1$ with Applications in Astrophysics

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Received December 10, 2015; revised September 27, 2016; published online November 15, 2016

Abstract. A Fredholm integral equation of the second kind in $L^1([a,b],C)$ with a weakly singular kernel is considered. Sufficient conditions are given for the existence and uniqueness of the solution. We adapt the product integration method proposed in $C^0([a,b],C)$ to apply it in $L^1([a,b],C)$, and discretize the equation. To improve the accuracy of the approximate solution, we use different iterative refinement schemes which we compare one to each other. Numerical evidence is given with an application in Astrophysics.

Keywords: Fredholm integral equation, product integration method, iterative refinement, Kolmogorov-Riesz-Fréchet theorem.

AMS Subject Classification: 45B05; 45A05; 45E99.

1 Introduction

We consider a Banach space $X$. Let $T$ be the integral operator defined by

$$\forall x \in X, \forall s \in [a,b], \quad T x(s) := \int_a^b L(s,t) H(s,t) x(t) dt,$$

(1.1)

where $(s,t) \mapsto H(s,t)$ is not smooth. For $z$ in the resolvent set of $T$, $\text{re}(T)$, and $y$ in $X$ we consider the Fredholm integral problem of the second kind

Find $\varphi \in X$ s.t. $(T - zI)\varphi = y,$

(1.2)

where $I$ denotes the identity operator on $X$.

To approximate the solution of this equation, we define a finite rank approximation $T_n$ of $T$, so that the approximate equation $(T_n - zI)\varphi_n = y$ or $(T_n - zI)\varphi_n = y_n$, where $y_n$ is an approximation of $y$, be uniquely solvable and the sequence of approximate solutions $\varphi_n$ converges to the exact solution $\varphi$ when $n$ tends to $+\infty$. 

Among them, different classes of methods rely on a sequence of projections \( \pi_n \) converging pointwise to the identity operator \( I \). For example the Galerkin operator is defined by \( T_n = \pi_n T \pi_n \), the projection operator by \( T_n = \pi_n T \), the Sloan operator by \( T_n = T \pi_n + \pi_n T - \pi_n T \pi_n \) (see [5], [10]). These approximations of \( T \) are all \( \nu \)-convergent to \( T \) (see [2]). This property ensures existence and uniqueness of \( \varphi_n \), and convergence to \( \varphi \).

In the case of the space \( X := C^0([a,b], \mathbb{C}) \) methods based upon numerical quadrature have been proposed, such as Nyström, truncated Nyström and subtraction of the singularity approximations (see [4]). In \( C^0([a,b], \mathbb{C}) \), we also encounter the so-called product integration method (see [5]). In this space, the assumptions are as follows:

**H1** \( L \in C^0([a,b] \times [a,b], \mathbb{C}) \).

**H2** \( H \) verifies:

\[
\begin{align*}
(H2.1) & \quad c_H := \sup_{s \in [a,b]} \int_a^b |H(s,t)| dt \text{ is finite}, \\
(H2.2) & \quad \lim_{h \to 0} \omega_H(h) = 0, \text{ where }
\omega_H(h) := \sup_{|s-\tau| \leq |h|, s, \tau \in [a,b]} \int_a^b |H(s,t) - H(\tau,t)| dt.
\end{align*}
\]

Let \( \Delta_n \), defined by

\[
a =: t_{n,0} < t_{n,1} < \ldots < t_{n,n} := b
\]

be a uniform grid of \([a,b]\). If \( h_n := (b-a)/n \), then \( t_{n,i} = a + ih_n \), for \( i = 0, 1, \ldots, n \). For \( x \in C^0([a,b], \mathbb{C}) \) and \( s \in [a,b] \), the linear interpolation scheme is given by

\[
[L(s,t)x(t)]_n := \frac{1}{h_n} [(t_{n,i}-t)L(s,t_{n,i-1})x(t_{n,i-1}) + (t-t_{n,i-1})L(s,t_{n,i})x(t_{n,i})]
\]

for \( i = 1, \ldots, n \) and \( t \in [t_{n,i-1}, t_{n,i}] \).

\( T_n \) is defined by replacing \( L(s,t)x(t) \) with \( [L(s,t)x(t)]_n \) in (1.1). In this method \( T_n \) is a bounded finite rank linear operator defined in \( C^0([a,b], \mathbb{C}) \) and hence it is compact.

Under hypotheses (H1) and (H2), for \( z \in \text{re}(T) \) and for \( n \) large enough, \( T_n - zI \) is invertible and its inverse is uniformly bounded, (see [5]).

In this paper we extend the product integration method to the space \( X := L^1([a,b], \mathbb{C}) \). It will appear that the properties of the method in \( C^0([a,b], \mathbb{C}) \) are preserved in \( L^1([a,b], \mathbb{C}) \). In Section 2, we present our method and we prove the existence and uniqueness of the approximate solution and its convergence to the exact solution. Section 3 is devoted to the numerical implementation of our algorithm. The choice of the integer \( n \) is limited by the capacity of the computer. The linear system to be solved is of the order of \( n \). So, it is
interesting to improve the accuracy of the approximate solution by applying some iterative refinement schemes. Section 4 is devoted to these schemes. In Section 5, we test our approximation with an academic example. In Section 6, we apply our method to a problem belonging to Astrophysics. Our method is compared with the projection method proposed by Titaud in [1] and [11].

2 The product integration method in $L^1([a, b], \mathbb{C})$

We use the following notations: the norm in $L^1([a, b], \mathbb{C})$ is denoted by $\|x\|_1 := \int_a^b |x(s)|ds$. The subordinated operator norm is also denoted by $\|\|_1$.

The oscillation of a function $x$ in $L^1([a, b], \mathbb{C})$, relatively to a parameter $h$ is defined by

$$w_1(x, h) := \sup_{|u| \leq |h|} \int_a^b |x(v + u) - x(v)|dv,$$

(2.1)

where $x$ is extended by 0 outside $[a, b]$.

The modulus of continuity of a continuous function on $[a, b]$ is defined as

$$w(x, h) := \sup_{u, v \in [a, b], |u - v| \leq h} |x(u) - x(v)|.$$

The modulus of continuity of a continuous function on $[a, b] \times [a, b]$ is defined as

$$w_2(f, h) := \sup_{u, v \in [a, b]^2, |u - v| \leq |h|} |f(u) - f(v)|.$$

If $x \in L^1([a, b], \mathbb{C})$, then $\lim_{h \to 0} w_1(x, h) = 0$. If $x \in C^0([a, b], \mathbb{C})$, then $\lim_{h \to 0} w(x, h) = 0$. If $f \in C^0([a, b]^2, \mathbb{C})$, then $\lim_{h \to 0} w_2(f, h) = 0$.

The aim of this section is to define the approximate operator $T_n$. The approximate solution of (1.2) will be, if it exists and is unique, the solution $\varphi_n$ of

$$(T_n - zI)\varphi_n = y.$$

(2.2)

$T_n$ is constructed so that $\varphi_n \to \varphi$. It is well known that a collectively compact convergence of $T_n$ towards $T$ guarantees the convergence of $\varphi_n$ towards $\varphi$.

Let us recall the collectively compact convergence:

DEFINITION 1. $T_n$ and $T$ are bounded linear operators from $X$ into $X$. The pointwise convergence, denoted by $T_n \xrightarrow{p} T$, means that

$$\forall x \in X, \|T_n x - Tx\| \to 0.$$ 

The collectively compact convergence is denoted by $T_n \xrightarrow{cc} T$ : if $T$ is compact

$$T_n \xrightarrow{p} T$$

and for some positive integer $n_0$ the set

$$W := \bigcup_{n \geq n_0} \{T_n x : x \in X, \|x\| \leq 1\}$$

is relatively compact in $X$. 

We begin by proving that $T$ is a compact bounded linear operator from $L^1([a, b], \mathbb{C})$ into itself. Then we propose an approximate operator $T_n$ which is a collectively compact convergent to $T$. Endly, we give an error estimation for the approximate solution in terms of the kernel, the norm of the exact solution, its oscillation in $L^1([a, b], \mathbb{C})$ and the mesh size.

The proof of the compactness in $L^1([a, b], \mathbb{C})$ relies on the Kolmogorov-Riesz-Fréchet theorem which is recalled here below. As usual, if $A$ is a set of functions, we define

$$A|_{\Omega} := \{f|_{\Omega} : f \in A\},$$

where $f|_{\Omega}$ is the restriction of $f$ to the subdomain $\Omega$.

**Theorem 1. (Kolmogorov-Riesz-Fréchet Theorem)** Let $F$ be a bounded set in $L^p(\mathbb{R}^q, \mathbb{C})$, $1 \leq p < \infty$. If

$$\lim_{\|h\| \to 0} \|\tau_h f - f\|_p = 0 \quad (2.3)$$

uniformly in $f \in F$, where $\tau_h f := f(\cdot + h)$, then the closure of $F|_{\Omega}$ is compact in $L^p(\Omega, \mathbb{C})$ for any measurable set $\Omega \subset \mathbb{R}^q$ with finite measure.

**Proof.** See [7]. As one finds a lot of different versions of this theorem in the litterature, we propose a proof of it in the Appendix in the case $q = 1$, $p = 1$ and $\Omega = [a, b]$. □

Now, the assumptions are as follows:

(P1) $L \in C_0^0([a, b] \times [a, b], \mathbb{C})$. Let

$$c_L := \sup_{(s,t) \in [a,b]^2} |L(s,t)|.$$

(P2) $H$ verifies:

(P2.1) $c_H := \sup_{t \in [a,b]} \int_a^b |H(s, t)| ds$ is finite.

(P2.2) $\lim_{h \to 0} w_H(h) = 0$,

where

$$w_H(h) := \sup_{t \in [a,b]} \int_a^b |\tilde{H}(s+h, t) - \tilde{H}(s, t)| ds$$

and

$$\tilde{H}(s, t) := \begin{cases} H(s, t), & \text{for } s \in [a,b], \\ 0, & \text{for } s \notin [a,b]. \end{cases}$$

**Lemma 1.**

$$\lim_{h \to 0^+} \epsilon(H, h) = 0,$$

where

$$\epsilon(H, h) := \sup_{t \in [a,b]} \int_{b-h}^b |H(s, t)| ds.$$
Proof. For $h > 0$, 
\[
0 \leq \int_{b-h}^{b} |H(s,t)| ds \leq \int_{b-h}^{b} |\tilde{H}(s+h,t) - \tilde{H}(s,t)| ds \\
\leq \int_{a}^{b} |\tilde{H}(s+h,t) - \tilde{H}(s,t)| ds \leq w_H(h).
\]

According to the assumption (P2.2), \( \sup_{t \in [a,b]} \int_{b-h}^{b} |H(s,t)| ds \to 0 \) as \( h \to 0^+ \).

This ends the proof. \( \Box \)

**Theorem 2.** Under the assumptions (P1) and (P2), the operator \( T \) is linear from \( L^1([a,b], \mathbb{C}) \) into itself and compact in \( L^1([a,b], \mathbb{C}) \).

**Proof.** For all \( x \in L^1([a,b], \mathbb{C}) \),
\[
\|Tx\|_1 = \int_{a}^{b} |\int_{a}^{b} L(s,t)H(s,t)x(t) dt| ds \leq \int_{a}^{b} \int_{a}^{b} |L(s,t)||H(s,t)||x(t)| dt ds \\
\leq c_L \int_{a}^{b} |x(t)| \int_{a}^{b} |H(s,t)| ds dt \leq c_L c_H \|x\|_1,
\]
so \( T \) is defined from \( L^1([a,b], \mathbb{C}) \) into itself.

The proof of the compactness of \( T \) relies on the Kolmogorov-Riesz-Fréchet theorem where \( p = 1, q = 1 \) and \( \Omega = [a,b] \). We introduce the operator \( \tilde{T} \):
\[
\tilde{T} x(s) := \begin{cases} 
Tx(s), & \text{for } s \in [a,b], \\
0, & \text{for } s \notin [a,b].
\end{cases}
\]

Let \( A \) and \( S \) be the following subsets of \( L^1(\mathbb{R}, \mathbb{C}) \) and \( L^1([a,b], \mathbb{C}) \) respectively:
\[
A := \{ \tilde{T} x : x \in L^1([a,b], \mathbb{C}), \|x\|_1 \leq 1 \}, \\
S := \{ Tx : x \in L^1([a,b], \mathbb{C}), \|x\|_1 \leq 1 \}.
\]

\( A \) is a bounded subset of \( L^1(\mathbb{R}, \mathbb{C}) \). Indeed
\[
\|\tilde{T} x\|_1 = \|Tx\|_1 \leq c_L c_H \|x\|_1 \leq c_L c_H.
\]

Let us prove that \( \lim_{h \to 0} \|\tau_h f - f\|_1 = 0 \) uniformly in \( f \in A \). For \( h > 0 \),
\[
\|\tau_h \tilde{T} x - \tilde{T} x\|_1 = \int_{a}^{b} |\tilde{T} x(s + h) - \tilde{T} x(s)| ds \\
= \int_{a}^{b-h} |T x(s + h) - T x(s)| ds + \int_{b-h}^{b} |T x(s)| ds.
\]

Hence
\[
\int_{b-h}^{b} |T x(s)| ds = \int_{b-h}^{b} \left| \int_{a}^{b} L(s,t)H(s,t)x(t) dt \right| ds \\
\leq c_L \|x\|_1 \epsilon(H, h) \leq c_L \epsilon(H, h)
\]
Lemma 2. For $i = 1, \ldots, n,$
\[
\int_a^b |w_{n,i}(s)|ds \leq h_n c_L c_H. \tag{2.6}
\]

For $h \in \mathbb{R}^+$,
\[
\int_{b-h}^b |w_{n,i}(s)|ds \leq h_n c_L \epsilon(H, h), \tag{2.7}
\]
\[
\int_a^{b-h} |w_{n,i}(s + h) - w_{n,i}(s)|ds \leq h_n c_H w_2(L, h) + h_n c_L w_H(h). \tag{2.8}
\]
Proof. For $t \in [t_{n,i-1}, t_{n,i}]$,

$$Q_n(1, s, t) = \frac{1}{h_n} [(t_{n,i} - t)L(s, t_{n,i-1}) + (t - t_{n,i-1})L(s, t_{n,i})],$$

$$|Q_n(1, s, t)| \leq \frac{c_n}{h_n} [t_{n,i} - t | + | t - t_{n,i-1}] = c_L.$$  

Hence, by Fubini’s theorem

$$\int_a^b |w_{n,i}(s)|ds \leq c_L \int_a^b \int_{t_{n,i-1}}^{t_{n,i}} |H(s, t)| dt ds \leq c_L h_n c_H$$

$$\int_{b-h}^b |w_{n,i}(s)|ds \leq c_L \int_{b-h}^b \int_{t_{n,i-1}}^{t_{n,i}} |H(s, t)| dt ds \leq c_L h_n \epsilon(H, h).$$

Also

$$\int_a^{b-h} |w_{n,i}(s + h) - w_{n,i}(s)|ds \leq \int_a^{b-h} \left| \int_{t_{n,i-1}}^{t_{n,i}} Q_n(1, s + h, t)H(s + h, t) - Q_n(1, s, t)H(s + h, t) dt \right| ds$$

$$- Q_n(1, s, t)H(s + h, t) dt ds \leq \int_a^{b-h} \int_{t_{n,i-1}}^{t_{n,i}} |Q_n(1, s + h, t) - Q_n(1, s, t)|H(s + h, t) dt ds$$

$$- H(s + h, t) dt ds \leq h_n w_2(L, h) \sup_{t \in [a, b]} \int_a^b |H(s, t)| ds$$

$$+ c_L h_n \sup_{t \in [a, b]} \int_a^{b-h} \left| \int_{t_{n,i-1}}^{t_{n,i}} \tilde{H}(s + h, t) - \tilde{H}(s, t)| ds \leq h_n c_H w_2(L, h) + h_n c_L w_H(h).$$

This ends the proof. □

Lemma 3. For $x \in L^1([a, b], \mathbb{C})$,

$$\sum_{i=1}^n \int_{t_{n,i-1}}^{t_{n,i}} |x(u) - c_{n,i}|du \leq 2w_1(x, h_n),$$

where $w_1(x, h_n)$ is defined by (2.1). For $t \in [a, b]$,

$$|Q_n(1, s, t) - L(s, t)| \leq w_2(L, h_n).$$

Proof. For $i = 1, \ldots, n$,

$$\int_{t_{n,i-1}}^{t_{n,i}} |x(u) - c_{n,i}|du \leq \frac{1}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} \int_{t_{n,i-1}}^{t_{n,i}} |x(u) - x(v)|dvdu$$

$$= \frac{2}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} \int_{t_{n,i-1}}^{t_{n,i}} |x(u) - x(v)|dudv.$$
Due to (2.6) in Lemma 2, for $T$ and the proof is complete. 

\[ \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} |x(u) - c_{n,i}| du \leq \frac{2}{h_n} \int_{0}^{h_n} \int_{a}^{b} |x(t + v) - x(v)| dv dt \]

Hence

\[ \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} |x(u) - c_{n,i}| du \leq 2 \sup_{\tau \in [0,h_n]} \int_{t_{n,i-1}}^{t_{n,i}} |x(t + v) - x(v)| dv dt \]

For $i = 1, \ldots, n$ and $t \in [t_{n,i-1}, t_{n,i}]$,

\[ |Q_n(1, s, t) - L(s, t)| \leq \frac{1}{h_n} [(t_{n,i} - t)(|L(s, t_{n,i-1}) - L(s, t)|) + (t - t_{n,i-1})(|L(s, t_{n,i}) - L(s, t)|)] \]

\[ \leq \sup_{s \in [a, b]} w(L(s, \cdot)) \frac{1}{h_n} [(t_{n,i} - t) + (t - t_{n,i-1})] \leq w_2(L, h_n) \]

and the proof is complete. \( \Box \)

**Theorem 3.** \( T_n \) is a compact linear operator from \( L^1([a,b], \mathbb{C}) \) into itself and \( T_n \xrightarrow{cc} T \).

**Proof.** Due to (2.6) in Lemma 2, for $x \in L^1([a,b], \mathbb{C}), \|T_n x\|_1 \leq c_L c_H \|x\|_1$ so $T_n$ is a linear bounded operator from $L^1([a,b], \mathbb{C})$ into itself. As $T_n$ is a linear bounded operator of finite rank, it is compact. Let us prove that $T_n \xrightarrow{p} T$. Lemma 3 implies that

\[ \|T_n x - T x\|_1 = \int_{a}^{b} \left| \sum_{i=1}^{n} c_{n,i} \int_{t_{n,i-1}}^{t_{n,i}} Q_n(1, s, t) H(s, t) dt \right| ds \]

\[ - \int_{a}^{b} L(s, t) H(s, t) x(t) dt | ds \]

\[ = \int_{a}^{b} \left| \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} \left( c_{n,i} Q_n(1, s, t) - L(s, t) x(t) \right) H(s, t) dt \right| ds \]

\[ = \int_{a}^{b} \left| \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} \left( (Q_n(1, s, t) - L(s, t)) x(t) \right. \right. \]

\[ + Q_n(1, s, t) \left( c_{n,i} - x(t) \right) \left) \right) H(s, t) dt \right| ds \]

\[ \leq \int_{a}^{b} \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} |Q_n(1, s, t) - L(s, t)| |x(t)| |H(s, t)| dt ds \]

\[ + \int_{a}^{b} \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} |Q_n(1, s, t)| |c_{n,i} - x(t)| |H(s, t)| dt ds \]
So we have $T_n \xrightarrow{p} T$. To prove the relatively compactness of

$$S_n := \{T_n x : n \geq 1, x \in L^1([a, b], \mathbb{C}), \|x\|_1 \leq 1\}$$

we follow the same scheme as in the proof of the compactness of $T$. We define the operator

$$\tilde{T}_n x(s) := \begin{cases} T_n x(s), & \text{for } s \in [a, b], \\ 0, & \text{for } s \notin [a, b], \end{cases}$$

and $A_n$ as the following subset of $L^1(\mathbb{R}, \mathbb{C})$

$$A_n := \{\tilde{T}_n x : x \in L^1([a, b], \mathbb{C}), \|x\|_1 \leq 1\}.$$  

$A_n$ is a bounded subset of $L^1(\mathbb{R}, \mathbb{C})$. Indeed,

$$\|\tilde{T}_n x\|_1 = \|T_n x\|_1 \leq c_L c_H \|x\|_1 \leq c_L c_H.$$  

Let us prove that $\lim_{h \to 0} \|\tau_h f - f\|_1 = 0$ uniformly in $f \in A_n$. For $h > 0$,

$$\|\tau_h \tilde{T}_n x - \tilde{T}_n x\|_1 = \int_a^b |\tilde{T}_n x(s + h) - \tilde{T}_n x(s)| ds = \int_a^{b-h} |T_n x(s + h) - T_n x(s)| ds + \int_{b-h}^b |T_n x(s)| ds.$$  

Hence, by (2.7) in Lemma 2,

$$\int_{b-h}^b |T_n x(s)| ds \leq \sum_{i=1}^n |c_{n,i}| \int_{b-h}^b |w_{n,i}(s)| ds \leq \frac{1}{h_n} \|x\|_1 h_n c_L \epsilon(H, h) \leq c_L \epsilon(H, h)$$

and because of (2.8) in Lemma 2,

$$\int_a^{b-h} |T_n x(s + h) - T_n x(s)| ds \leq \sum_{i=1}^n |c_{n,i}| |w_{n,i}(s + h) - w_{n,i}(s)| ds \leq \sum_{i=1}^n \left|c_{n,i}\left(h_n c_H w_2(L, h) + h_n c_L w_H(h)\right)\right| \leq \|x\|_1 (c_H w_2(L, h) + c_L w_H(h)) \leq c_H w_2(L, h) + c_L w_H(h).$$
Hence
\[ \|\tau_h \tilde{T}_n x - \tilde{T}_n x\|_1 \leq \|x\|_1 (c_H w_2(L, h) + c_L w_H(h) + c_L e(H, h)). \tag{2.10} \]
For h < 0, we have similar bounds. Then \( \|\tau_h f - f\|_1 \to 0 \) as \( h \to 0 \) uniformly in \( f \in A_n \). From the Kolmogorov-Riesz-Fréchet theorem, \( A_n |_{[a, b]} \) is relatively compact so \( T_n \xrightarrow{cc} T \).

**Proposition 1.** Let \( z \in \text{re}(T) \). For \( n \) large enough, \( T_n - zI \) is invertible and it exists a positive number \( c_z > 0 \) such that
\[ \|(T_n - zI)^{-1}\|_1 \leq c_z. \tag{2.11} \]

**Proof.** It is a consequence of the collectively compact convergence (see [3]). \( \Box \)

**Theorem 4.** For \( z \in \text{re}(T) \) and under hypotheses (P1) and (P2), for \( n \) large enough, the approximate operator equation (2.2) has a unique solution \( \varphi_n \) satisfying the following error bound:
\[ \|\varphi - \varphi_n\|_1 \leq c_z c_H (\|\varphi\|_1 w_2(L, h_n) + 2 c_L w_1(\varphi, h_n)). \]

**Proof.** According to (2.9) in the proof of Theorem 3,
\[ \|\varphi - \varphi_n\| \leq \|(T_n - zI)^{-1}\|_1 \|(T - T_n)\varphi\|_1 \leq c_z c_H (\|\varphi\|_1 w_2(L, h_n) + 2 c_L w_1(\varphi, h_n)), \]
which ends the proof. \( \Box \)

**Remark 1.** Often in practice, the kernel \( H \) is of convolution type. Let us fix \( a = 0 \) and \( b = 1 \). We suppose that there is a function \( g \) such that
\[ H(s, t) = g(|s - t|), \]
where \( g \) is a weakly singular function defined on \([0, 1]\). This means that \( g \) satisfies the following properties:
\[ \lim_{s \to 0^+} g(s) = +\infty, \quad g \in C^0([0, 1], \mathbb{R}) \cap L^1([0, 1], \mathbb{R}), \]
\[ g \geq 0 \text{ and } g \text{ is a decreasing function in } [0, 1]. \]

**Proposition 2.** When the factor \( H \) in the kernel of the operator \( T \) is of weakly singular convolution type, then \( H \) verifies all the conditions imposed by the product integration methods.

**Proof.**
\[(H2.1) \forall s \in [0, 1], \text{ we have}
\[ \int_0^1 g(|s - t|) dt = \int_0^s g(s - t) dt + \int_s^1 g(t - s) dt = \int_0^s g(\tau) d\tau + \int_0^{1-s} g(\tau) d\tau \leq 2 \int_0^1 g(\tau) d\tau < +\infty. \]
(P2.1) is also valid because the variables \( s \) and \( t \) play symmetric roles.

(H2.2) Let us prove that, for \( h > 0 \),

\[
\lim_{h \to 0^+} \sup_{|s - \tau| \leq h} \int_0^1 |g(|s - t|) - g(|\tau - t|)|dt = 0.
\]

Let \( \psi \) be the function defined by \( t \mapsto \psi(t) = |g(|s - t|) - g(|\tau - t|)| \).

Suppose that \( \tau < s \). It is easy to prove that \( \psi \) has an axial symmetry with respect to \( \xi = s + \tau/2 \) over the interval \([\tau, s]\). Let \( G(t) := \int_0^t g(s)ds \). Then

\[
\int_0^1 \psi(t)dt = \int_0^\tau \psi(t)dt + \int_\tau^\xi \psi(t)dt + \int_\xi^s \psi(t)dt + \int_s^1 \psi(t)dt
\]

\[
= \int_0^\tau g(\tau - t) - g(s - t)dt + 2 \int_\tau^\xi g(t - \tau) - g(s - t)dt
\]

\[
+ \int_s^1 g(t - s) - g(t - \tau)dt
\]

\[
= G(\tau) - G(s) + G(s - \tau) + 2G(s - \tau) - 2G(s - \tau)
\]

\[
+ 2G(s - \tau) + G(1 - s) + G(s - \tau) - G(1 - \tau)
\]

\[
= 4 \int_0^{s - \tau} g(\sigma)d\sigma - \int_\tau^s g(\sigma)d\sigma - \int_{1-s}^{1-\tau} g(\sigma)d\sigma \leq 4 \int_0^{s - \tau} g(\sigma)d\sigma,
\]

hence,

\[
\omega_H(h) \leq \sup_{|s - \tau| \leq h} \int_0^1 |g(|s - t|) - g(|\tau - t|)|dt \leq 4 \int_0^{\frac{h}{2}} g(\sigma)d\sigma,
\]

so

\[
\lim_{h \to 0^+} \omega_H(h) = 0.
\]

(P2.2) Let us prove that, for \( h > 0 \),

\[
\lim_{h \to 0^+} \sup_{t \in [0,1]} \int_0^1 |\tilde{g}(|s + h - t|) - \tilde{g}(|s - t|)|ds = 0.
\]

For \( t \in [0,1] \),

\[
\int_0^1 |\tilde{g}(|s + h - t|) - \tilde{g}(|s - t|)|ds = \int_0^1 |\tilde{g}(|t - h - s|) - \tilde{g}(|t - s|)|ds,
\]

\[
\leq \omega_H(h),
\]

so

\[
\lim_{h \to 0^+} w_H(h) = \lim_{h \to 0^+} \int_0^1 |\tilde{g}(|s + h - t|) - \tilde{g}(|s - t|)|ds = 0,
\]

which ends the proof. □
3 Iterative refinement

Recall that \( z \neq 0 \) because \( T \) is compact and \( z \in \text{re}(T) \). Consider that the solution of (1.2) is approximated by \( G_n(z)y \), where \( G_n(z) \) is an approximate inverse of \( T - zI \). The accuracy of \( G_n(z)y \) may be improved using the following iterative refinement schemes:

\[
x_n^{(0)} := G_n(z)y, \quad x_n^{(k+1)} := x_n^{(0)} + (I - G_n(z)(T - zI))x_n^{(k)}.
\]

In [11], \( G_n(z) \) has been one of the following operators:

Scheme A (Atkinson):

\[
G_n(z) := R_n(z) := (T_n - zI)^{-1},
\]

Scheme B (Brakhage):

\[
G_n(z) := \frac{1}{z}R_n(z)(T_n - T),
\]

Scheme C (Titaud):

\[
G_n(z) := \frac{1}{z}(TR_n(z) - I).
\]

Their convergence properties and error bounds have already been studied in terms of \( T, T_n \) and \( R_n(z) \) (see [11] pp 40-41). If \( \varphi \) is the solution of (1.2), Scheme A (Atkinson):

\[
\|x_n^{(k)} - \varphi\|_1/\|\varphi\|_1 \leq \|(R_n(z)(T_n - T))^{k+1}\|_1,
\]

Scheme B (Brakhage):

\[
\|x_n^{(k)} - \varphi\|_1/\|\varphi\|_1 \leq \left(\frac{1}{z}R_n(z)(T_n - T)T\right)^{k+1}\|_1,
\]

Scheme C (Titaud):

\[
\|x_n^{(k)} - \varphi\|_1/\|\varphi\|_1 \leq \left(\frac{1}{z}TR_n(z)(T_n - T)\right)^{k+1}\|_1.
\]

Let us state error estimations for these three refinement schemes for the approximate operator \( T_n \) defined by (2.5) in this paper.

**Theorem 5.** For \( T_n \) defined by (2.5), the following error bounds are satisfied:

**Scheme A (Atkinson):**

\[
\|x_n^{(2\ell-1)} - \varphi\|_1/\|\varphi\|_1 \leq m_z^\ell \mathcal{E}(h_n)^\ell,
\]

\[
\|x_n^{(2\ell)} - \varphi\|_1/\|\varphi\|_1 \leq 2d_zc_Hc_Lm_z^\ell \mathcal{E}(h_n)^\ell,
\]

**Scheme B (Brakhage):**

\[
\|x_n^{(2\ell-1)} - \varphi\|_1/\|\varphi\|_1 \leq \left(d_z/z\right)^{2\ell} \mathcal{E}(h_n)^{2\ell},
\]

\[
\|x_n^{(2\ell)} - \varphi\|_1/\|\varphi\|_1 \leq \left(d_z/z\right)^{2\ell+1} \mathcal{E}(h_n)^{2\ell+1},
\]

Scheme C (Titaud):

\[
\|x_n^{(2\ell-1)} - \varphi\|_1/\|\varphi\|_1 \leq (d_z/z)^{2\ell} (2c_H^2 c_L^2) \mathcal{E}(h_n)^{2\ell-1},
\]
\[
\|x_n^{(2\ell)} - \varphi\|_1/\|\varphi\|_1 \leq (d_zz)^{2\ell+1} (2c_H^2 c_L^2) \mathcal{E}(h_n)^{2\ell},
\]

where

\[
\mathcal{E}(h_n) := 3c_H^2 c_L w_2(L, h_n) + 2c_H c_L^2 w_H(h_n) + 2c_H^2 \epsilon(H, h_n),
\]
\[m_z := 2d_z^2 + 2c_H c_L d_z^3, \quad d_z := \max(c_z, \|R(z)\|).
\]

Proof. Using (2.9),

\[
\|(T - T_n)Tx\| \leq c_H\|Tx\|w_2(L, h_n) + 2c_H c_L\|x\|_1(c_L w_H(h_n))
\]
\[
+ c_H w_2(L, h_n)\|T\|\|x\| + 2c_H c_L w_1(Tx, h_n).
\]

As

\[
w_1(Tx, h_n) = \sup_{[u] \in [0, h_n]} \|\tau_u \tilde{T}x - \tilde{T}x\|_1
\]

and due to (2.4),

\[
\|(T - T_n)Tx\| \leq c_H\|Tx\|w_2(L, h_n) + 2c_H c_L\|x\|_1(c_L w_H(h_n))
\]
\[
+ c_H w_2(L, h_n)\|T\|\|x\| + 2c_H c_L w_1(Tx, h_n).
\]

As

\[
\|T_n x\|_1 \leq c_L c_H\|x\|_1, \quad w_1(T_n x, h_n) = \sup_{[u] \in [0, h_n]} \|\tau_u \tilde{T}_n x - \tilde{T}_n x\|_1
\]

and because of (2.10),

\[
\|(T - T_n)T_n x\| \leq c_H^2 c_L w_2(L, h_n)\|x\|_1
\]
\[
+ 2c_H c_L\|x\|_1(c_L w_H(h_n) + c_H w_2(L, h_n) + c_L \epsilon(H, h_n))
\]
\[
\leq \|x\|_1(3c_H^2 c_L w_2(L, h_n) + 2c_H c_L^2 w_H(h_n) + 2c_H^2 \epsilon(H, h_n)) \leq \|x\|_1 \mathcal{E}(h_n).
\]

- Scheme A. As

\[
(T_n - T)R_n(z)T = (T_n - T)R_n(z)(T - T_n)TR(z) + (T_n - T)TR(z)
\]

and according to (2.11),

\[
\|(R_n(z)(T_n - T))^2\| = \|(R_n(z)(T_n - T)R_n(z)T_n + R_n(z)(T_n - T)R_n(z)T)\|
\]
\[
= \|(R_n(z)(T_n - T)T_n R_n(z) + R_n(z)(T_n - T)R_n(z)T)\|
\]
\[
\leq c_z\|(T_n - T)T_n\| + c_z\|(T_n - T)R_n(z)(T - T_n)TR(z) + (T_n - T)TR(z)\|.
\]
We have
\[
\|(R_n(z)(T_n - T))^2\| \leq d_z^2\|T_n - T\| + 2c_HC_Ld_z^3\|(T - T_n)T\|
+ d_z^2\|(T_n - T)T\| \leq (2d_z^2 + 2c_HC_Ld_z^3)\mathcal{E}(h_n) \leq m_z\mathcal{E}(h_n).
\]
Then
\[
\|(R_n(z)(T_n - T))^{2\ell}\|_1 \leq m_z^\ell\mathcal{E}(h_n)^\ell,
\]
so
\[
\frac{\|x_n^{(2\ell - 1)} - \varphi\|_1}{\|\varphi\|_1} \leq m_z^\ell\mathcal{E}(h_n)^\ell, \quad \frac{\|x_n^{(2\ell)} - \varphi\|_1}{\|\varphi\|_1} \leq 2d_zc_HC_Lm_z^\ell\mathcal{E}(h_n)^\ell.
\]

- **Scheme B.** As
\[
\|(\frac{1}{z}R_n(z)(T_n - T)T)^{2\ell}\|_1 \leq \left(\frac{d_z}{z}\right)^{2\ell}\mathcal{E}(h_n)^{2\ell},
\]
then
\[
\frac{\|x_n^{(2\ell - 1)} - \varphi\|_1}{\|\varphi\|_1} \leq \left(\frac{d_z}{z}\right)^{2\ell}\mathcal{E}(h_n)^{2\ell}, \quad \frac{\|x_n^{(2\ell)} - \varphi\|_1}{\|\varphi\|_1} \leq \left(\frac{d_z}{z}\right)^{2\ell+1}\mathcal{E}(h_n)^{2\ell+1}.
\]

- **Scheme C.** As
\[
\left(\frac{1}{z}TR_n(z)(T_n - T)\right)^{k+1} = \left(\frac{1}{z}\right)^{k+1}TR_n(z)((T_n - T)TR_n(z))^k(T_n - T),
\]
\[
\|(\frac{1}{z}TR_n(z)(T_n - T))^{k+1}\|_1 \leq \left(\frac{d_z}{z}\right)^{k+1}(2c_HC_L^2)\|(T_n - T)T\|_1^k
\leq \left(\frac{d_z}{z}\right)^{k+1}(2c_HC_L^2)\mathcal{E}(h_n),
\]
so
\[
\|x_n^{(2\ell - 1)} - \varphi\|_1/\|\varphi\|_1 \leq \left(\frac{d_z}{z}\right)^{2\ell}(2c_HC_L^2)\mathcal{E}(h_n)^{2\ell-1},
\]
\[
\|x_n^{(2\ell)} - \varphi\|_1/\|\varphi\|_1 \leq \left(\frac{d_z}{z}\right)^{2\ell+1}(2c_HC_L^2)\mathcal{E}(h_n)^{2\ell}.
\]
This concludes the proof. □

**Remark 2.** The upperbound of Scheme B appears to be the optimal one among the three error bounds. It improves slightly upon the one of Scheme C and is twice better than the one of Scheme A.

### 4 Numerical Implementations

The approximate equation is
\[
T_n\varphi_n - z\varphi_n = y, \text{ i.e.}
\]
\[
\forall s \in [a, b], \quad \sum_{i=1}^n w_{n,j}(s) \frac{1}{h_n} \int_{t_{n,j-1}}^{t_{n,j}} \varphi_n(u)du - z\varphi_n(s) = y(s).
\]

By calculating the average over \([t_{n,i-1}, t_{n,i}], i = 1, \ldots, n\), of each member of the equation, we obtain a linear system of the form \((A - zI)x = d\), where

\[
A(i, j) := \frac{1}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} w_{n,j}(s) ds, \quad i, j = 1, \ldots, n,
\]

\[
d(i) := \frac{1}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} y(s) ds, \quad i = 1, \ldots, n,
\]

\[
x(i) := \frac{1}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} \varphi_n(s) ds, \quad i = 1, \ldots, n.
\] (4.1)

After solving the linear system, the approximate solution can be written as

\[
\varphi_n(s) = \frac{1}{z} \left( \sum_{i=1}^{n} w_{n,j}(s) x(i) - y(s) \right).
\]

To measure the quality of the approximation we calculate the relative residual

\[
r(\varphi_n) := \frac{\| (T - zI) \varphi_n - y \|_1}{\| y \|_1}.
\]

In practice the evaluation of \(T\) is often not possible, so we replace it with \(T_m\) where \(m \gg n\) and we calculate the average over \([t_{m,i-1}, t_{m,i}], i = 1, \ldots, m\), of \((T - zI)\varphi_n - y\) and of \(y\). We obtain two vectors of size \(m\), and we calculate the vector norm in \((\mathbb{C}^m, \| \cdot \|_1)\).

5 Numerical Illustration

As an academic example we have taken

\[
- \int_0^1 \ln(|s - t|) \varphi(t) dt - \varphi(s) = y(s),
\]

with unique solution \(\varphi(s) = s^2\). The estimations of the relative residual with \(m = 100\) for two methods: the projection method proposed by Titaud in [11] and the \(L^1([a,b], \mathbb{C})\) product integration method are shown in Table 1. We observe that the \(L^1([a,b], \mathbb{C})\) product integration method is faster than the projection method.

<table>
<thead>
<tr>
<th>(n)</th>
<th>Projection method</th>
<th>Product integration method</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.0968</td>
<td>0.0246</td>
</tr>
<tr>
<td>20</td>
<td>0.0499</td>
<td>0.0087</td>
</tr>
<tr>
<td>50</td>
<td>0.0211</td>
<td>0.0018</td>
</tr>
</tbody>
</table>

Figure 1 shows the profile of the matrix \(A\) defined by (4.1). It is a full matrix.

In Figure 2 we chose \(n = 100, m = 1000\) for a relative residual tolerance of \(10^{-12}\). We note that Scheme B is the fastest one to reach the tolerance.

The theoretical Remark 2 of Section 3 is confirmed by this numerical experiment.
An Extension of the Product Integration Method to $L^1$

Figure 1. Matrix A of the academic illustration.

Figure 2. Residual convergence with the three refinement schemes of the academic illustration.

6 An Application in Astrophysics

The radiative transfer problem is a system of differential equations coupled with a Fredholm integral equation of the second kind. It describes the energy conserved by a beam radiation traveling, such that a beam of radiation can lose or gain energy through absorbing, scattering and emitting medium. Let $\tau_*$ be the optical width of the medium, (see [8]). An example of this equation is

$$\frac{\varpi(s)}{2} \int_0^{\tau_*} E_1(|s-t|)\varphi(t)dt - \varphi(s) = y(s),$$

where $E_1$ is the first integral exponential function:

$$\forall \nu \geq 1, \quad E_\nu(s) := \int_0^1 \frac{e^{-s/\mu}}{\mu^{2-\nu}} d\mu$$

and the function $\varpi$ describes the albedo. In our numerical example $\varpi(s) = 0.7 \exp(-s)$ and

$$y(s) = \begin{cases} 
-0.3, & \text{for } s \in [0,50[, \\
0, & \text{for } s \in [50,100].
\end{cases}$$

The singularity of that example is different from the Cauchy singularity treated by Beltram with the product integration method in [6].

Figure 3. Matrix $A$ of the Astrophysics application.

Figure 3 shows the profile of the matrix $A$ defined by (4.1). It is a sparse matrix.

The relative residual associated to the approximate solution $\varphi_n$ obtained by the projection method and the product integration method proposed in this paper are shown in Table 2. We observe that the product integration method converges faster than the projection method.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Projection method</th>
<th>Product integration method</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.0267</td>
<td>0.0172</td>
</tr>
<tr>
<td>20</td>
<td>0.0252</td>
<td>0.0145</td>
</tr>
<tr>
<td>50</td>
<td>0.0151</td>
<td>0.0075</td>
</tr>
</tbody>
</table>

For large values of $n$ the computation of $\varphi_n$ is prohibitively costly so that we will use the refinement schemes introduced in Section 3 to compute the final approximate solution. In Figure 4 we chose $n = 100$, $m = 1000$ for a relative residual tolerance of $10^{-12}$. We note that Scheme C is the fastest one to reach the tolerance. This confirms the results obtained in [9].

Remark 3. In this application, Scheme C is apparently faster than Scheme B. This could be explained by the difference between the profiles of the corresponding auxiliary matrices $A$ (see Figure 1 and Figure 3).

Acknowledgements

This research has been partially supported by the Indo French Center for Applied Mathematics (IFCAM).
Figure 4. Residual convergence with the three refinement schemes in the Astrophysics application.

References


Appendix

Proof of the Kolmogorov-Riesz-Fréchet theorem. Without loss of generality we prove the theorem for the case $p = 1, q = 1$ and $\Omega = [a, b]$. To simplify the notation, $\| \cdot \|_1$ denotes the norm in $L^1(\Omega, \mathbb{C})$ and also the norm in $L^1(\mathbb{R}, \mathbb{C})$. $\| \cdot \|_\infty$ denotes the norm in $C^0(\Omega, \mathbb{C})$ and also the norm in $C^0(\mathbb{R}, \mathbb{C})$.

As $L^1(\Omega, \mathbb{C})$ is a complete space, we just need to prove that $\mathcal{F}|_{\Omega}$ is precompact i.e.: For any $\varepsilon > 0$ there exist functions $f_1, f_2, \ldots, f_N \in L^1(\Omega, \mathbb{C})$ such that

$$\mathcal{F}|_{\Omega} \subset \bigcup_{i=1}^N B_1(f_i, \varepsilon),$$

where $B_1(f_i, \varepsilon)$ denotes the open ball in $L^1(\Omega, \mathbb{C})$ centered in $f_i$ and of radius $\varepsilon$. The proof consists in constructing the functions $f_i$. The main idea of the proof is to apply a convolution regularization process to deal with continuous functions and to be able to apply the Arzela-Ascoli theorem.

Step 1: Regularization process

Let us consider the regularizing sequence defined by

$$\rho_n(x) := n\rho(nx),$$

where

$$\rho(x) := \begin{cases} 
  k \exp\left(-\frac{1}{1-x^2}\right), & \text{for } |x| \leq 1, \\
  0, & \text{otherwise,}
\end{cases}$$

and $k$ is a constant such that $\|\rho\|_1 = 1$. For all $n \in \mathbb{N}$, $\rho_n$ is infinitely differentiable. If $*$ denotes the convolution product, and if $f \in L^1(\mathbb{R}, \mathbb{C})$, $\rho_n * f$ is a regularization of $f$ in the sense that it is smooth: $\rho_n * f$ is infinitely differentiable. We know that $\rho_n * f \in L^1(\mathbb{R}, \mathbb{C})$ and also $\rho_n * f \rightarrow f$ in $L^1(\mathbb{R}, \mathbb{C})$. We prove a stronger result under assumption (2.3):

$$\rho_n * f \rightarrow f$$

uniformly in $f \in \mathcal{F}$ in $L^1(\mathbb{R}, \mathbb{C})$.

$$|\rho_n * f(x) - f(x)| \leq \int_{-1/n}^{1/n} |f(x-y) - f(x)| \rho_n(y) dy,$$

so that for all $f \in \mathcal{F}$,

$$\int_{\mathbb{R}} |\rho_n * f(x) - f(x)| dx \leq \int_{-1/n}^{1/n} \int_{-1/n}^{1/n} |f(x-y) - f(x)| \rho_n(y) dx dy$$

$$= \int_{-1/n}^{1/n} \rho_n(y) \left( \int_{\mathbb{R}} |f(x-y) - f(x)| dx \right) dy$$

$$\leq \int_{-1/n}^{1/n} \rho_n(y) dy \sup_{|y| \leq \frac{1}{n}} \|\tau_y f - f\|_1 = \sup_{|y| \leq \frac{1}{n}} \|\tau_y f - f\|_1.$$
Hence for all $f \in \mathcal{F}$,

$$\|\rho_n * f - f\|_1 \leq \sup_{|y| \leq 1/n} \|\tau_y f - f\|_1.$$ 

According to assumption (2.3), for all $\varepsilon > 0$, $\exists N_0 \in \mathbb{N}$:

$$n \geq N_0 \Rightarrow \|\rho_n * f - f\|_1 \leq \varepsilon,$$

for all $f \in \mathcal{F}$.

**Step 2: Application of Arzela-Ascoli theorem to $H_n := \{\rho_n * f : f \in \mathcal{F}\}|_{\Omega}$**

Here $n$ is fixed. Due to the regularization properties, $H_n$ is a subset of $C^0(\Omega, \mathbb{C})$. Let us prove that $H_n$ is bounded in $C^0(\Omega, \mathbb{C})$ equipped with the infinity norm $\|\cdot\|_{\infty}$. As $\mathcal{F}$ is bounded in $L^1(\mathbb{R}, \mathbb{C})$,

$$\|\rho_n * f\|_{\infty} \leq \|\rho_n\|_{\infty}\|f\|_1 \leq M\|\rho_n\|_{\infty},$$

where $M := \sup_{f \in \mathcal{F}} \|f\|_1$. Let us prove that $H_n$ is equicontinuous.

Let $x_1, x_2 \in \omega$. 

$$|\rho_n * f(x_1) - \rho_n * f(x_2)| = \left| \int (\rho_n(x_1 - y) - \rho(x_2 - y)) f(y) dy \right|$$

$$\leq \int |\rho_n(x_1 - y) - \rho(x_2 - y)||f(y)| dy$$

$$\leq \|\nabla \rho_n\|_{\infty}|x_1 - x_2|\|f\|_1 \leq M\|\nabla \rho_n\|_{\infty}|x_1 - x_2|,$$

where $\nabla \rho_n$ is the gradient of $\rho_n$. According to Arzela-Ascoli theorem, $H_n$ is relatively compact in $C^0(\Omega, \mathbb{C})$ so it is precompact.

**Step 3: Construction of the functions $f_i$**

As $H_n$ is precompact, for $\varepsilon > 0$ there exist functions $f_i \in C^0(\Omega, \mathbb{C}), i = 1, \ldots, N$, such that $H_n \subset \cup_{i=1}^{N} B_{\infty}(f_i, \varepsilon)$, where $B_{\infty}(f_i, \varepsilon)$ denotes the ball in $C^0(\Omega, \mathbb{C})$ centered in $f_i$ and of radius $\varepsilon$, i.e:

$$\forall \rho_n * f \in H_n, \exists f_i \in C^0(\Omega, \mathbb{C}) : \|\rho_n * f - f_i\|_{\infty} < \varepsilon.$$ 

**Step 4: Conclusion**

Let us show that $\mathcal{F}|_{\Omega}$ is precompact. Let $\varepsilon > 0$ and $f \in \mathcal{F}|_{\Omega}$. According to the step 1, $\exists N_0 \in \mathbb{N}$:

$$n \geq N_0 \Rightarrow \|\rho_n * f - f\|_1 \leq \varepsilon,$$

for all $f \in \mathcal{F}$.

Let us fix $n \geq N_0$. According to the step 3, there exists $i \in \{1, \ldots, N\}$, such that $\|\rho_n * f - f_i\|_{\infty} < \varepsilon$. We have

$$\|f - f_i\|_1 \leq \|\rho_n * f - f\|_1 + \|\rho_n * f - f_i\|_1,$$

$$\|\rho_n * f - f_i\|_1 = \left( \int_a^b |\rho_n * f(x) - f_i(x)| dx \right)$$

$$\leq (b - a)\|\rho_n * f - f_i\|_{\infty} < (b - a)\varepsilon.$$ 

Hence

$$\|f - f_i\|_1 \leq (1 + b - a)\varepsilon.$$ 

So $\mathcal{F}|_{\Omega} \subset \cup_{i=1}^{N} B_1(f_i, (1 + b - a)\varepsilon)$ and $\mathcal{F}|_{\Omega}$ is relatively compact.