COEFFICIENT STABILITY OF OPERATOR–DIFFERENCE SCHEMES

P.P. MATUS\(^1\), B.S. JOVANOVIĆ\(^2\)

\(^1\) Institute of Mathematics, NAS Belarus
Surganova str. 11, Minsk, Belarus

\(^2\) University of Belgrade, Faculty of Mathematics
Studentski trg 16, 11000 Belgrade, Yugoslavia

E-mail: \(^1\) matus@im.bas-net.by, \(^2\) bosko@matf.bg.ac.yu

Received August 24, 1999

ABSTRACT

A priori estimates expressing continuous dependence of the solution of a first order evolutionary equation in Hilbert space on initial condition, right hand side and operator perturbations are obtained in time–integral norms. Analogous results hold for corresponding finite difference schemes.

1. INTRODUCTION

While solving differential equations, one is often in a situation that the coefficients are not given exactly, but approximately (e.g. as a result of a physical measurement). Therefore, the problem of stability for the variable coefficients differential equation is of great importance (coefficient stability). Analogous problem is present in corresponding numerical methods which approximate the differential equation. In spite of importance of this problem a priori estimates expressing continuous dependence of the solution on the right hand side and operator perturbation (so called strong stability) till recent times have been obtained only for stationary problems (see \([2, 4, 5, 10, 12]\)). Strong stability of difference and operator–difference schemes were investigated in \([3, 7, 8, 9]\) and \([11]\). In these papers corresponding a priori estimates were obtained in uniform in time norms. Using integral in time norm \([7]\) we proved

---

\(^1\) Research supported by Belarusian Fund of Fundamental research, grant F98–019.

\(^2\) Research supported by Serbian Fund of Fundamental research, grant 01M03/C.
a priori estimates under very weak assumptions on the right hand side of the equation. We applied this approach investigating the accuracy of the solution of difference schemes for the problems with generalized solutions (see [3], [6], [9], [11]).

This paper deals with the construction of stability estimates for the first order evolutionary problems in Hilbert spaces, and corresponding finite difference schemes, in the case of perturbed operators, right hand sides and initial conditions. A priori error estimates in time–integral norms are obtained under some natural assumptions on perturbed operators.

2. STABILITY OF OPERATOR–DIFFERENTIAL SCHEMES

Let $H$ be a Hilbert space with the inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Let $A$ be a constant, self-adjoint, positive definite linear operator in $H$, i.e.

$$A(t) = A(0), \quad A = A^* \geq \delta E, \quad \delta = \text{const} > 0,$$

where $E$ is the unit operator in $H$. We denote $H_D$, where $D = D^* > 0$, the space with the following inner product and norm

$$(y, v)_D = (Dy, v), \quad \|y\|_D = \sqrt{(Dy, y)}.$$

We consider the Cauchy problem

$$\frac{du}{dt} + Au = f(t), \quad 0 < t < T; \quad u(0) = u_0, \quad (2.1)$$

where $u_0$ is a given element of $H$, $f(t)$ is a given function and $u(t)$ is an unknown function with values in $H$.

Using the energy method and Fourier expansion we can prove the following assertion.

Lemma 2.1. The solution of the problem (2.1) satisfies a priori estimates:

$$\int_0^T \left( \|Au(t)\|^2 + \left\| \frac{du(t)}{dt} \right\|^2 \right) dt \leq 2 \|u_0\|_A^2 + 2 \int_0^T \|f(t)\|^2 dt, \quad (2.2)$$

$$\int_0^T \|u(t)\|^2_A dt + \int_0^T \int_0^T \frac{\|u(t) - u(t')\|^2}{|t - t'|^2} dt' dt \\
\leq (1 + 3\pi) \|u_0\|^2 + (1 + 4\pi) \int_0^T \|f(t)\|^2_{A^{-1}}, \quad (2.3)$$

$$\int_0^T \|u(t)\|^2 dt \leq \|u_0\|^2_{A^{-1}} + \int_0^T \|A^{-1}f(t)\|^2 dt. \quad (2.4)$$
Setting \( f(t) = dg(t)/dt \) in (2.1) one obtains the Cauchy problem in the form
\[
\frac{du}{dt} + Au = \frac{dg}{dt}, \quad 0 < t < T; \quad u(0) = u_0.
\] (2.5)
An analogous proposition holds for the problem (2.5).

**Lemma 2.2.** The solution of the problem (2.5) satisfies a priori estimates:
\[
\int_0^T ||u(t)||^2_A \, dt + \int_0^T \int_0^T \frac{||u(t) - u(t')||^2}{|t - t'|^2} \, dt \, dt' \\
\leq (\pi + 4 \pi^2) \left\{ ||u_0||^2 + \int_0^T \int_0^T \frac{||g(t) - g(t')||^2}{|t - t'|^2} \, dt \, dt' \right\} \\
+ \int_0^T \left( \frac{1}{t} + \frac{1}{T - t} \right) ||g(t)||^2 \, dt, \quad \text{(2.6)}
\]
\[
\int_0^T ||u(t)||^2 \, dt \leq ||u_0 - g(0)||^2_A + \int_0^T ||g(t)||^2 \, dt. \quad \text{(2.7)}
\]

**Example 2.1.** Let us consider the initial-boundary value problem for one-dimensional heat transfer equation
\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( k(x) \frac{\partial u}{\partial x} \right) + f(x, t), \quad (x, t) \in Q_T = (0, 1) \times (0, T), (2.8)
\]
\[
u(0, t) = u(1, t) = 0, \quad u(x, 0) = u_0(x).
\]
The problem (2.8) can be represented in the form (2.1), setting \( H = L_2(0, 1) \)
and \( Av = -\frac{d}{dx} \left( k(x) \frac{du}{dx} \right) \). If \( k \in C^1[0, 1] \) and \( k(x) \geq k_0 > 0 \) the operator \( A \)
maps the set \( D(A) = W^1_2(0, 1) \cap W^2_2(0, 1) \subset L_2(0, 1) \) on \( L_2(0, 1) \). The inverse operator \( A^{-1} \) is
\[
A^{-1}v(x) = -\int_0^x \frac{1}{k(x')} \int_0^{x'} v(x'') \, dx'' \, dx' \\
+ \left( \int_0^1 \frac{dx'}{k(x')} \right)^{-1} \left( \int_0^x \frac{dx'}{k(x')} \right) \left( \int_0^1 \frac{1}{k(x')} \int_0^{x'} v(x'') \, dx'' \, dx' \right). \quad \text{(2.9)}
\]
The following inequalities hold (see [6])
\[
c_1 ||v||^2_{W^2_2(0, 1)} \leq ||v||^2_A = \int_0^1 k(x) |v'(x)|^2 \, dx \leq c_2 ||v||^2_{W^2_2(0, 1)}, 
\]
\[
v \in W^1_2(0, 1),
\]
\[
c_3 ||v||_{W^2_2(0, 1)} \leq ||Av||_{L_2(0, 1)} \leq c_4 ||v||_{W^2_2(0, 1)},
\]
\[
v \in W^1_2(0, 1) \cap W^2_2(0, 1), \quad \text{(2.10)}
\]
In such a way, inequalities (2.2), (2.3) and (2.4) show the stability of the problem (2.8) in spaces $W^{2,1}_2(Q_T)$, $W^{1,1/2}_2(Q_T)$ and $L_2(Q_T)$.

3. STRONG STABILITY OF OPERATOR–DIFFERENTIAL SCHEMES

Along with (2.1) and (2.5) let us consider analogous Cauchy problems with perturbed right hand side, initial condition and operator:

$$
\frac{d\tilde{u}}{dt} + \tilde{A}\tilde{u} = \tilde{f}(t), \quad 0 < t < T; \quad \tilde{u}(0) = \tilde{u}_0, \tag{3.1}
$$

and

$$
\frac{d\tilde{u}}{dt} + \tilde{A}\tilde{u} = \frac{d\tilde{g}(t)}{dt}, \quad 0 < t < T; \quad \tilde{u}(0) = \tilde{u}_0. \tag{3.2}
$$

We deal with the problem of estimating the perturbation of the solution

$$
z(t) = \tilde{u}(t) - u(t)
$$

with the values of perturbations of $u_0$, $A$ and $f$ (or $g$). Let the perturbed operators satisfy the analogous assumptions as the operator $A$:

$$
\tilde{A}(t) = \tilde{A}(0), \quad \tilde{A} = \tilde{A}^* \geq \tilde{\delta}E, \quad \tilde{\delta} = \text{const} > 0.
$$

As the measure of perturbation of operator we shall use the positive constant $\alpha$ in inequalities

$$
\|(\tilde{A} - A)v\| \leq \alpha\|\tilde{A}v\|, \tag{3.3}
$$

$$
\|((\tilde{A} - A)v, v)\| \leq \alpha(Av, v), \tag{3.4}
$$

$$
\|A^{-1}\tilde{A}v - v\| \leq \alpha\|v\|. \tag{3.5}
$$

From (2.1) and (3.1), (i.e. (2.5) and (3.2)), one obtains

$$
\frac{dz}{dt} + Az = (\tilde{f}(t) - f(t)) - (\tilde{A} - A)\tilde{u}, \quad 0 < t < T, \quad z(0) = \tilde{u}_0 - u_0 \tag{3.6}
$$

or

$$
\frac{dz}{dt} + Az = \frac{d(\tilde{g} - g)}{dt} - (\tilde{A} - A)\tilde{u}, \quad 0 < t < T; \quad z(0) = \tilde{u}_0 - u_0. \tag{3.7}
$$

Using lemmas 2.1 and 2.2 and conditions (2.3) – (2.5) we obtain the following result.
Theorem 3.1. The perturbation of the solution of the problem (2.1) satisfies a priori estimates:

\[
\int_0^T \left( \|A\|_2 z(t)\|^2 + \left\| \frac{dz(t)}{dt} \right\| \right) dt \leq 2 \|\tilde{u}_0 - u_0\|_{A}^2 + 4 \int_0^T \|\tilde{f}(t)\|^2 dt + 4 \alpha^2 \left( \|\tilde{u}_0\|_{A}^2 + \int_0^T \|\tilde{f}(t)\|^2 dt \right),
\]

if the condition (3.3) holds;

\[
\int_0^T \|z(t)\|^2_{A} dt + \int_0^T \left( \|z(t) - z(t')\|^2 + \frac{\|z(t) - z(t')\|^2}{|t - t'|} \right) dt dt' \leq (2 + 15\pi) \left( \|\tilde{u}_0 - u_0\|_{A}^2 + 2 \int_0^T \|A^{-1}(\tilde{f}(t) - f(t))\|^2 dt \right)
\]

+ \int_0^T \|\tilde{f}(t) - f(t)\|^2_{A^{-1}} dt + \frac{\alpha^2}{1 - \alpha} \left( \|\tilde{u}_0\|_{A}^2 + \int_0^T \|\tilde{f}(t)\|^2_{A^{-1}} dt \right),
\]

if the condition (3.4) holds;

\[
\int_0^T \|\tilde{z}(t)\|^2 dt \leq \|\tilde{u}_0 - u_0\|_{A}^2 + 2 \int_0^T \|\tilde{f}(t) - f(t)\|^2 dt
\]

+ \int_0^T \|\tilde{f}(t)\|^2 dt + 2 \alpha^2 \left( \|\tilde{u}_0\|_{A}^2 + \int_0^T \|\tilde{f}(t)\|^2 dt \right),
\]

if the condition (3.5) holds;

The perturbation of the solution of the problem (2.5) satisfies a priori estimates:

\[
\int_0^T \|z(t)\|^2_{A} dt + \int_0^T \left( \|z(t) - z(t')\|^2 + \frac{\|z(t) - z(t')\|^2}{|t - t'|} \right) dt dt' \leq 34 \pi^2 \left( \|\tilde{u}_0 - u_0\|_{A}^2 + 2 \int_0^T \|\tilde{g}(t) - g(t)\|^2 dt + \frac{1}{\alpha^2} \left( \|\tilde{u}_0\|_{A}^2 + \int_0^T \|\tilde{g}(t)\|^2 dt \right) \right),
\]

if the condition (3.4) holds;

\[
\int_0^T \|\tilde{z}(t)\|^2 dt \leq 2 \|\tilde{u}_0 - u_0 - \tilde{g}(0)\|_{A}^2 + 2 \int_0^T \|\tilde{g}(t) - g(t)\|^2 dt
\]

+ \int_0^T \|\tilde{g}(t)\|^2 dt + 2 \alpha^2 \left( \|\tilde{u}_0 - g(0)\|_{A}^2 + \int_0^T \|\tilde{g}(t)\|^2 dt \right),
\]
if the condition (3.5) holds.

Example 3.1. Let us consider together with (2.8) a perturbed initial-boundary value problem

\[
\frac{\partial \tilde{u}}{\partial t} = \frac{\partial}{\partial x} \left( k(x) \frac{\partial \tilde{u}}{\partial x} \right) + f(x, t), \quad (x, t) \in Q_T = (0, 1) \times (0, T)
\]
\[
\tilde{u}(0, t) = \tilde{u}(1, t) = 0, \quad \tilde{u}(x, 0) = \tilde{u}_0(x).
\]

We have \( A\tilde{v} = -\frac{d}{dx} \left( k(x) \frac{dv}{dx} \right) \) and \( \tilde{A}\tilde{v} = -\frac{d}{dx} \left( \tilde{k}(x) \frac{d\tilde{v}}{dx} \right) \). In such a way

\[
((\tilde{A} - A)v, v)_{L^2(0, 1)} = \int_0^1 (\tilde{k}(x) - k(x)) |v'(x)|^2 \, dx.
\]

where from follows that the condition (3.4) is satisfied for

\[
|\tilde{k}(x) - k(x)| \leq \alpha k(x).
\]

Using (2.10) we get

\[
\|((\tilde{A} - A)v)\|_{L^2(0, 1)} \leq \|k - \tilde{k}\|_{C[0, 1]} \|v''\|_{L^2(0, 1)} + \|k' - \tilde{k}'\|_{C[0, 1]} \|v'\|_{L^2(0, 1)}
\]
\[
\leq \left( \|k - \tilde{k}\|_{C[0, 1]} + \|k' - \tilde{k}'\|_{C[0, 1]} \right) \|v\|_{W^2_0(0, 1)}
\]
\[
\leq \frac{1}{c_3} \left( \|k - \tilde{k}\|_{C[0, 1]} + \|k' - \tilde{k}'\|_{C[0, 1]} \right) \|Av\|_{L^2(0, 1)}.
\]

Consequently, if

\[
|\tilde{k}(x) - k(x)| \leq \alpha_1 \quad \text{and} \quad |\tilde{k}'(x) - k'(x)| \leq \alpha_2,
\]

then (3.3) holds with \( \alpha = (\alpha_1 + \alpha_2)/c_3 \).

It follows from (2.9) that

\[
A^{-1} \tilde{A}v(x) - v(x) = - \int_0^x \left[ \frac{\tilde{k}'(x') - k'(x')}{k(x')} + \frac{k'(x') (k(x') - \tilde{k}(x'))}{k^2(x')} \right] \int_0^1 \frac{dx'}{k(x')} + \frac{k'(x') (k(x') - \tilde{k}(x'))}{k^2(x')} \int_0^1 \frac{dx'}{k(x')} \int_0^1 \frac{dx''}{k(x'')}
\]
\[
+ \frac{k'(x') (k(x') - \tilde{k}(x'))}{k^2(x')} \int_0^1 \frac{dx''}{k(x'')} \int_0^1 \frac{dx'''}{k(x''')} \int_0^1 \frac{dx'''}{k(x''')} \int_0^1 \frac{dx''''}{k(x'''')} v(x') \, dx' + \frac{\tilde{k}(x) - k(x)}{k(x)} v(x).
\]

Hence, when inequalities (3.9) are satisfied, then (3.5) holds with \( \alpha = \alpha_1 (k_0 + 2 \|k'\|_{C[0, 1]}/k_0^2 + 2\alpha_2/k_0) \).
4. STABILITY OF OPERATOR–DIFFERENCE SCHEMES

Analogous results hold for operator–difference schemes. Let \( H_h \) be a finite dimensional Hilbert space, with inner product \((\cdot, \cdot)_h\) and norm \(\| \cdot \|_h\). Let \( A_h \) be a constant, self-adjoint, positive definite linear operator in \( H_h \), i.e.

\[
A_h(t) = A_h(0), \quad A_h = A_h^* \geq \delta_h E_h, \quad \delta_h = \text{const} > 0,
\]

where \( E_h \) is the unit operator in \( H_h \). We denote \( H_{D_h} \), where \( D_h = D_h^* > 0 \), the space with the following inner product and norm

\[
(y, v)_{D_h} = (D_h y, v)_h, \quad \| y \|_{D_h} = \sqrt{(D_h y, y)_h}.
\]

Let \( \omega_r \) be an uniform mesh on \((0, T)\) with the step \( \tau = T/m, \omega_r = \omega_r \cup \{0\} \) and \( \Delta_r = \omega_r \cup \{0, T\} \). We will use the standard notation of the theory of difference schemes [1].

We consider the two-level weighted finite difference scheme

\[
v_k + A_h v_{(\sigma)} = \varphi(t), \quad t \in \omega_r^-; \quad v(0) = v_0, \quad (4.1)
\]

where \( v_{(\sigma)} = \sigma \tilde{v} + (1 - \sigma)v = \sigma v(t + \tau) + (1 - \sigma)v(t), \sigma \geq (1 + \varepsilon)/2 - 1/(\tau \| A_h \|) \) is the weight parameter, \( v_0 \) is a given element of \( H_h \), \( \varphi(t) \) is a given function and \( v(t) \) is unknown function with values in \( H_h \). We also consider the finite difference scheme

\[
v_k + A_h v_{(\sigma)} = \psi_k, \quad t \in \omega_r^-; \quad v(0) = v_0, \quad (4.2)
\]

where \( \psi(t) \) is a given mesh function with values in \( H_h \).

The following analogous of lemmas 2.1 and 2.2 hold.

**Lemma 4.1.** The solution of the finite difference scheme (4.1) satisfies a priori estimates:

\[
\tau \sum_{t \in \omega_r} \| A_h v(t) \|_h^2 + \tau \sum_{t \in \omega_r} \| v(t) \|_h^2 \leq C \left( \| v_0 \|_{A_h}^2 + \tau \| A_h v_0 \|_h^2 + \tau \sum_{t \in \omega_r} \| \varphi(t) \|_h^2 \right),
\]

\[
\tau \sum_{t \in \omega_r} \| v(t) \|_{A_h}^2 + \tau \sum_{t \in \omega_r} \sum'_{t' \in \omega_r, t \neq t'} \frac{\| v(t) - v(t') \|_h^2}{|t - t'|^2} \leq C \left( \| v_0 \|_{A_h}^2 + \tau \| v_0 \|_{A_h}^2 + \tau \sum_{t \in \omega_r} \| \varphi(t) \|_{A_h^{-1}}^2 \right).
\]

\[
\tau \sum_{t \in \omega_r} \| v(t) \|_h^2 \leq C \left( \| v_0 \|_{A_h^{-1}}^2 + \tau \| v_0 \|_h^2 + \tau \sum_{t \in \omega_r} \| A_h^{-1} \varphi(t) \|_h^2 \right).
\]
Here we denoted \( \sum'_{t \in \Delta_r} w(t) = \frac{w(0)}{2} + \sum_{t \in \omega} w(t) + \frac{w(T)}{2} \).

**Lemma 4.2.** The solution of the finite difference scheme \((4.2)\) satisfies a priori estimates:

\[
\tau \sum'_{t \in \Delta_r} \|v(t)\|_{A_h}^2 + \tau^2 \sum'_{t \in \Delta_r, t' \in \Delta_r, t \neq t'} \frac{\|v(t) - v(t')\|_{A_h}^2}{|t - t'|^2} \leq C \left\{ \|v_0\|_{A_h}^2 + \tau \|v_0\|_{A_h}^2 \right\}
\]

\[
+ \tau^2 \sum'_{t \in \Delta_r, t' \in \Delta_r, t \neq t'} \frac{\|\psi(t) - \psi(t')\|_{A_h}^2}{|t - t'|^2} + \tau \sum_{t \in \omega_r} \left( \frac{1}{t} + \frac{1}{T - t} \right) \|\psi(t)\|_{A_h}^2 \right},
\]

\[
\tau \sum_{t \in \omega_r} \|v(t)\|_{A_h}^2 \leq C \left\{ \|v_0 - \psi(0)\|_{A_h}^2 + \tau \|v_0\|_{A_h}^2 + \tau \sum_{t \in \omega_r} \|\psi(t)\|_{A_h}^2 \right\}.
\]

**Example 4.1.** Let \( \omega_h \) be an uniform mesh in \((0, 1)\) with the step \( h = 1/n \), \( \omega^- = \omega_h \cup \{0\}, \omega^+ = \omega_h \cup \{0, 1\} \) and \( \omega_{h+} = \omega_h \times \omega_r \). We approximate the initial-boundary value problem \((2.8)\) with the standard weighted difference scheme

\[
v_t = \left( a v_x^r \right)_x + f, \quad x \in \omega_h, \quad t \in \omega^-; \\
v(0, t) = v(1, t), \quad t \in \omega_r; \quad v(x, 0) = u_0(x), \quad x \in \omega_h.
\]

Here \( a \) is some stencil functional of \( k \), e.g. \( a(x) = k(x + h/2) \).

Let \( H_h \) be the set of functions defined on the mesh \( \omega_h \), which vanish in the nodes \( x = 0 \) and \( x = 1 \), \( (v, w)_h = h \sum_{x \in \omega_h} v(x) w(x) \) and \( A_h w = -(a w_x)_x \). The characteristics of the operator \( A_h \) are well known [1, p. 120], in particular \( A_h = A_h^* \geq 8 k_0 E_h \). In such a way, the difference scheme \((4.3)\) can be represented in the form of operator-difference scheme \((4.1)\). The inverse operator \( A_h^{-1} \) is

\[
A_h^{-1} w(x) = \sum_{x' = 0}^{x-h} a(x') \sum_{x'' = 0}^{x'} h w(x'')
\]

\[
+ \left( \sum_{x' = 0}^{x-h} a(x') \right)^{-1} \left( \sum_{x' = 0}^{x-h} a(x') \right) \left( \sum_{x'' = 0}^{x'} a(x'') \right) h w(x'),
\]

The following inequalities hold

\[
k_0 \|w_r\|_{A_h}^2 \leq \|w\|_{A_h}^2 \leq k_1 \|w_r\|_{A_h}^2,
\]

where \( \|w\|_{A_h}^2 = h \sum_{x \in \omega^-} w^2(x) \), and

\[
k_2 \|w_{r,x}\|_{A_h}^2 \leq \|A_h w\|_{A_h}^2 \leq k_3 \|w_{r,x}\|_{A_h}^2.
\]
5. STRONG STABILITY OF OPERATOR–DIFFERENCE SCHEMES

Along with (4.1) and (4.2) let us consider analogous finite difference schemes with perturbed right hand sides, initial conditions and operators:

\[ \hat{v}_t + \hat{A}_h \hat{v}^{(\sigma)} = \hat{\varphi}(t), \quad t \in \omega^-; \quad \hat{v}(0) = \hat{v}_0 \]

(5.1)

and

\[ \tilde{v}_t + \tilde{A}_h \tilde{v}^{(\sigma)} = \tilde{\psi}(t), \quad t \in \omega^-; \quad \tilde{v}(0) = \tilde{v}_0. \]

(5.2)

We deal with the problem of estimating the perturbation of the solution

\[ z(t) = \hat{v}(t) - v(t) \]

with perturbations of \( v_0, A_h \) and \( \varphi \) (or \( \psi \)).

Let the perturbed operator satisfy the analogous assumptions as the operator \( A_h \):

\[ \hat{A}_h(t) = \hat{A}_h(0), \quad \tilde{A}_h = \tilde{A}_h^* \geq \delta_h E_h, \quad \delta_h = \text{const} > 0. \]

As the measure of perturbation of operator we shall use the positive constant \( \alpha \) in inequalities

\[ \|(\hat{A}_h - A_h)w\|_h \leq \alpha \|\hat{A}_h w\|_h, \]

(5.3)

\[ \|(\hat{A}_h - A_h)w, w\|_h \leq \alpha (A_h w, w)_h, \]

(5.4)

\[ \|A_h^{-1}\hat{A}_h w - w\|_h \leq \alpha \|w\|_h. \]

(5.5)

From (4.1) and (5.1), consequently (4.2) and (5.2), we obtain the following difference schemes for the perturbation of the solution

\[ z_t + A_h z^{(\sigma)} = (\hat{\varphi}(t) - \varphi(t)) - (\hat{A}_h - A_h) \hat{v}^{(\sigma)}, \quad t \in \omega^-; \quad z(0) = \hat{v}_0 - v_0 \]

(5.6)

and

\[ z_t + A_h z^{(\sigma)} = (\hat{\psi}(t) - \psi(t)) - (\tilde{A}_h - A_h) \tilde{v}^{(\sigma)}, \quad t \in \omega^-; \quad z(0) = \tilde{v}_0 - v_0. \]

(5.7)

The following analog of the theorem 3.1 holds.

**Theorem 5.1.** The perturbation of the solution of difference scheme (4.1) satisfies a priori estimates:

\[
\tau \sum_{t \in \omega^+} \|A_h z(t)\|_h^2 + \tau \sum_{t \in \omega^+} \|z(t)\|_h^2 \\
\leq C \left\{ \|\hat{v}_0 - v_0\|_{A_h}^2 + \tau \|A_h (\hat{v}_0 - v_0)\|_{A_h}^2 \\
+ \tau \sum_{t \in \omega^-} \|\hat{\varphi}(t) - \varphi(t)\|_h^2 + \alpha^2 \left( \|\hat{v}_0\|_{A_h}^2 + \tau \sum_{t \in \omega^-} \|\hat{\varphi}(t)\|_h^2 \right) \right\},
\]

where \( C \) is a constant independent of \( \tau \) and \( h \).
if the condition (5.3) holds:

\[
\tau \sum_{t \in \mathcal{T}_{\tau}} \|z(t)\|_{A_h}^2 + \tau^2 \sum_{t \in \mathcal{T}_{\tau}, t' \in \mathcal{T}_{\tau}, t' \neq t} \frac{\|z(t) - z(t')\|_h^2}{|t - t'|^2} \leq C \left\{ \|\tilde{v}_0 - v_0\|_h^2 + \tau \|\tilde{v}_0 - v_0\|_h^2 \right\},
\]

\[
+ \tau \|\tilde{v}_0 - v_0\|_{A_h}^2 + \tau \sum_{t \in \mathcal{T}_{\tau}} \|\varphi(t) - \varphi(t)\|_{A_h}^2 + \alpha^2 \left( \|\tilde{v}_0\|_{A_h}^2 + \tau \sum_{t \in \mathcal{T}_{\tau}} \|\tilde{v}(t)\|_{A_h}^2 \right) \}.
\]

if the condition (5.4) holds:

\[
\tau \sum_{t \in \mathcal{T}_{\tau}} \|z(t)\|_{A_h}^2 \leq C \left\{ \|\tilde{v}_0 - v_0\|_{A_h}^2 + \tau \|\tilde{v}_0 - v_0\|_{A_h}^2 \right\},
\]

\[
+ \tau \sum_{t \in \mathcal{T}_{\tau}} \|A_h^{-1}(\varphi(t) - \varphi(t))\|_{A_h}^2 + \alpha^2 \left( \|\tilde{v}_0\|_{A_h}^2 + \tau \sum_{t \in \mathcal{T}_{\tau}} \|A_h^{-1}\tilde{\varphi}(t)\|_{A_h}^2 \right) \}.
\]

if the condition (5.5) holds.

The perturbation of the solution of difference scheme (4.2) satisfies a priori estimates:

\[
\tau \sum_{t \in \mathcal{T}_{\tau}} \|z(t)\|_{A_h}^2 + \tau^2 \sum_{t \in \mathcal{T}_{\tau}, t' \in \mathcal{T}_{\tau}, t' \neq t} \frac{\|z(t) - z(t')\|_h^2}{|t - t'|^2} \leq C \left\{ \|\tilde{v}_0 - v_0\|_h^2 + \tau \|\tilde{v}_0 - v_0\|_h^2 \right\},
\]

\[
+ \tau \|\tilde{v}_0 - v_0\|_{A_h}^2 + \tau \sum_{t \in \mathcal{T}_{\tau}} \|\varphi(t) - \varphi(t)\|_{A_h}^2 + \alpha^2 \left( \|\tilde{v}_0\|_{A_h}^2 + \tau \sum_{t \in \mathcal{T}_{\tau}} \|\tilde{v}(t)\|_{A_h}^2 \right) \}.
\]

if the condition (5.4) holds:

\[
\tau \sum_{t \in \mathcal{T}_{\tau}} \|z(t)\|_{A_h}^2 \leq C \left\{ \|\tilde{v}_0 - v_0 - [\tilde{\varphi}(0) - \varphi(0)]\|_{A_h}^2 + \tau \|\tilde{v}_0 - v_0\|_h^2 \right\},
\]

\[
+ \tau \sum_{t \in \mathcal{T}_{\tau}} \|\tilde{\varphi}(t) - \varphi(t)\|_{A_h}^2 + \alpha^2 \left( \|\tilde{v}_0 - \tilde{\varphi}(0)\|_{A_h}^2 + \tau \sum_{t \in \mathcal{T}_{\tau}} \|\tilde{\varphi}(t)\|_{A_h}^2 \right) \}.
\]

if the condition (5.5) holds.

Example 5.1. Let us consider, together with (4.3), a perturbed finite difference scheme

\[
\tilde{v}_t = \left( \tilde{a}_x \tilde{v}_x \right)_x + \tilde{f}, \quad x \in \omega_h, \ t \in \omega_{\tau} ;
\]

\[
\tilde{v}(0, t) = \tilde{v}(1, t), \quad t \in \omega_{\tau} ;
\]

\[
\tilde{v}(x, 0) = \tilde{u}_0(x), \quad x \in \omega_h.
\]
Let us set $A_h w = -(a w_x)_x$ and $\tilde{A}_h w = -(\tilde{a} w_x)_x$. From (4.5) follows

$$(\tilde{A}_h - A_h) w, w)_h = h \sum_{x \in \omega_h} (\tilde{a}(x) - a(x)) w^2_{x}(x),$$

wherefrom we can see, that the condition (5.4) is satisfied when

$$|\tilde{a}(x) - a(x)| \leq \alpha a(x).$$

Using (4.6) we obtain

$$\|\tilde{A}_h - A_h| w\|_h \leq \|\tilde{a} - a\|_{C;h} \|w_x\|_h + \|\tilde{a}_x - a_x\|_{C;h} \|w_x\|_h$$

$$\leq \frac{1}{k_2} \left( \|\tilde{a} - a\|_{C;h} + \frac{1}{8} \|\tilde{a}_x - a_x\|_{C;h} \right) \|A_h w\|_h,$$

where $\| \cdot \|_{C;h}$ is the mesh $C$-norm. In such a way, if

$$|\tilde{a}(x) - a(x)| \leq \alpha_1 \quad \text{and} \quad |\tilde{a}_x(x) - a_x(x)| \leq \alpha_2,$$

then (5.3) holds with $\alpha = \alpha_1 / k_2 + \alpha_2 / (8 k_2)$.

The condition (5.9) is also sufficient for the inequality (5.5). Using (4.4) we obtain (5.5) with $\alpha = \alpha_1 (k_0 + 2 \|a_x\|_{C;h}) / k_0^2 + 2 \alpha_2 / k_0$.

REFERENCES

P. P. Matus, B. S. Jovanović


OPERATORINIŲ-DIFERENCIALINIŲ SCHEMU KOEFICIENTINIS STABILUMAS

P. P. MATUS, B. S. JOVANOVIĆ